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B.Sc. Sem-5, Mathematics, Analysis-1: BSCC506A

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Although we are used to notion of sets and function from our school time. We'll still recall some important definition and properties of sets and function in this section.

A *set* is a "collection", "family" or "class" (of objects, numbers, things, other sets etc.)
Examples of a set:

- A collection of all pens in this room.
- A collection of all the chairs in this building
- A collection of all the tables and chairs in this building
- The class of all natural numbers

Example of a collection which is not a set.

- A collection of all the best actors of India

Some useful notations:

- $x \in A$, the element x is in the set A or x belongs to A
- If x is *not* in A, we write $x \notin A$
- If every element of a set A also belongs to a set B, we say that A is a subset of B and write $A \subseteq B$ (or simply $A \subset B$)or B is a superset of A and write $B \supseteq A$
- A set A is a called a proper subset of a set B if $A \subseteq B$, but there is at least one element of B that is not in A. we write $A \subsetneq B$
- Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$.

- A set can be defined either by listing all its members (e.g. $\{1, 2, 3\}$, $\{2, 4, 4, 7\}$)
  or by specifying some unique property that determine the elements of the set.
  e.g. $\mathbb{N} = \{1, 2, 3 \dots\}$ = the set of all natural numbers,
  $\text{A} = \{n \in \mathbb{N} : n^2 - 4 = 0\}$
- ϕ denotes the *empty set*, the unique set which contains no elements. Sometimes it is also written $\{\}$.

Some more examples of sets we generally use in mathematics

- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of Integers
- $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}$, the set of all rational numbers
- $\mathbb{R}$, the set of real numbers
- $\mathbb{C}$, the set of complex numbers
- We can write the set of even natural numbers as $\{n \in \mathbb{N} : n \textit{ is even}\}$ or $\{2n : n \in \mathbb{N}\}$

Set operations

- The *union* of sets A and B is the set
  $A \cup B = \{x : x \in A \text{ or } x \in B\}$
  e.g. $A = \{1, 3, 5\}$ and $B = \{1, 2, 4\}$ then $A \cup B = \{1, 2, 3, 4, 5\}$
- *The intersection of sets A and B is the set*
  $A \cap B = \{x : x \in A \text{ and } x \in B\}$
  e.g. $A = \{1, 3, 5\}$ and $B = \{1, 2, 4\}$ then $A \cap B = \{1\}$
- *The complement of B relative to A (or A minus B) is the set*
  $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$
  e.g. $A = \{1, 3, 5\}$ and $B = \{1, 2, 4\}$ then $A \setminus B = \{3, 5\}$
- *Two sets A and B are said to be disjoint if they have no elements in common. (i.e. $A \cap B = \phi$)*

Theorem 1

If A, B and C are sets, then

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof

We shall prove part (a). To show that the set on the right hand side of the equation is same as the set on the left hand side of the equation, we have to pick an element form LHS and show that the element is in RHS set too. And vice versa (i.e. to pick an element form RHS and show that the element is in LHS set too)

Let $x \in A \setminus (B \cup C)$.
This implies $x \in A$ but $x \notin (B \cup C)$.
Which implies $x \in A$ but x is neither in B nor x is in C.
This implies x is in A but not in B and x is in A but not in C
This implies $x \in (A \setminus B)$ and $x \in (A \setminus C)$
This implies $x \in (A \setminus B) \cap (A \setminus C)$
This means $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$ (i)

Now we'll prove the converse.
Let $x \in (A \setminus B) \cap (A \setminus C)$.
This implies $x \in A$ but $x \notin B$ and $x \in A$ but $x \in C$.
This implies $x \in A$ but x is neither in B nor in C.
This implies $x \in A$ but $x \notin (B \cup C)$
This implies $x \in A \setminus (B \cup C)$.
This shows $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$ (ii)

By (i), (ii) and definition of equality of sets we can say that
$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
Part (b) of the theorem can also be proved by similar technique.

- If $\{A_1, A_2, A_3, \ldots, A_n\}$ are sets then their union is denoted by $\bigcup\limits_{i=1}^{n} A_i$ and their intersection is denoted by $\bigcap\limits_{i=1}^{n} A_i$.
- Similarly, for $\{A_1, A_2, A_3, \ldots\}$ are countably many sets then their union is denoted by $\bigcup\limits_{i=1}^{\infty} A_i$ and their intersection is denoted by $\bigcap\limits_{i=1}^{\infty} A_i$.

Definition 1

If A and B are nonempty sets, then the *Cartesian product* $A \times B$ *of* A *and* B *is the set of all ordered pairs* (a, b) *with* $a \in A$ *and* $b \in B$. *That is,*

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

- If $A = \{1, 2, 3\}$ and $B = \{1, 5\}$, then

$$A \times B = \{(1, 1), \quad (1, 5), \quad (2, 1), \quad (2, 5), \quad (3, 1), \quad (3, 5)\}$$

Definition 2

Let A and B be sets. Then a function from A to B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

The set A of first elements of a *function* f *is called the* *domain* *of* f *and is often denoted by* $D(f)$. *The set of all second elements in* f *is called the* *range* *of* f *and is of ten denoted by* $R(f)$.

Note that, although $D(f) = A$, we only have $R(f) \subseteq B$.

Definition 3

Let $f : A \to B$ be a function from A to B

(a) The function f is said to be injective (or to be one-one) if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. If f is an *injective function, we also say that f is an injection.*

(b) The function f is said to be surjective (or to map A onto B) if $f(A) = B$; that is, if the range $R(f) = B$. If f is a surjective function, we also say that f is a surjection.

(c) If f is both injective and surjective, then f is said to be bijective. If f is bijective, we also say that f is a bijection.

(d) If $f : A \to B$ is a bijection of A onto B, then

$$g = \{(b, a) \in B \times A : (a, b) \in f\}$$

is a function on B into A. This function is called the inverse function of f, and is denoted by f^{-1}. The function f^{-1} is also called the inverse of f

Definition 4

If $f : A \to B$ and $g : B \to C$, and if $R(f) \subseteq D(g) = B$, then the *composite function* $g \circ f$ *is the function from* A *into* C *defined by*

$$(g \circ f)(x) = g(f(x)) \quad \textit{for all} \quad x \in A$$

Finite and Infinite sets

- The empty set ϕ is said to have 0 elements.
- A set S is said to have n elements ($n \in \mathbb{N}$) if there exists a bijection from the set $\{1, 2, 3, \ldots, n\}$ (we call it $\mathbb{N}_n$) onto S.
- A set S is said to be *finite if it is either empty or it has n elements for some $n \in \mathbb{N}$.*
- *A set S is said to be infinite if it is not finite.*

Theorem 2 (Uniqueness Theorem)

If S is a finite set, then the number of elements in S is a unique number in $\mathbb{N}$.

Theorem 3

The set $\mathbb{N}$ of natural numbers is an infinite set.

Theorem 4

(a) *If A is a set with m elements and B is a set with n elements and if $A \cap B = \phi$, then $A \cup B$ has $m+n$ elements.*

(b) *If A is a set with $m \in \mathbb{N}$ elements and $C \subseteq A$ is a set with 1 element, then $A \setminus C$ is a set with $m-1$ elements.*

(c) *If C is an infinite set and B is a finite set, then $C \setminus B$ is an infinite set.*

Proof:-

We shall prove part (a) of the theorem. Proof of part (b) and (c) are similar.

Let f be a bijection of $\mathbb{N}_m$ onto A, and let g be a bijection of $\mathbb{N}_n$ onto B. Now we define h on $\mathbb{N}_{m+n}$ by $h(i) = f(i)$ for $i = 1, \ldots, m$ and $h(i) = g(i - \ m)$ for $i = m+1, \ldots, m+n$. We will show that h is a bijection from $\mathbb{N}_{m+n}$ onto $A \cup B$.

(i) To show h is one-to-one function.
Let $i, j \in \{1, 2, \ldots, m, m+1, \ldots, m+n\}$ such that $i \neq j$.
w.l.g. we can also assume $i \leq j$.
We have 3 possibilities:
(1) $i, j \in \{1, 2, \ldots, m\}$
since $i, j \in \{1, 2, \ldots, m\}$, $h(i) = f(i)$ and $h(j) = f(j)$ also as $i \neq j$ implies $f(i) \neq f(j)$ (because f is one-to-one). Hence $h(i) \neq h(j)$.
(2) $i, j \in \{m+1, \ldots, m+n\}$
proof is similar to (i) and the fact that g is one-to -one
(3) $i \in \{1, 2, \ldots, m\}$ and $j \in \{m+1, \ldots, m+n\}$
since $i \in \{1, 2, \ldots, m\}$, implies $h(i) = f(i) \in A$.
and $j \in \{m+1, \ldots, m+n\}$, implies $h(j) = g(j) \in B$.
now, $A \cap B = \phi$. This implies $h(i) \neq h(j)$.
So in all the three cases $i \neq j$ implies $h(i) \neq h(j)$.
This proves h is one-to-one function.

Now we'll show that h is onto.
Let $x \in A \cup B$. since $A \cap B = \phi$, $x \in A$ or $x \in B$ but not in both.
If $x \in A$ means $\exists i \in \{1, 2, \ldots, m\}$, such that $f(i) = x$ ($\because f$ is onto).
Therefore $h(i) = f(i) = x$.
If $x \in B$ means $\exists i \in \{m+1, \ldots, m+n\}$, such that $g(i) = x$ ($\because g$ is onto).
Therefore $h(i) = g(i) = x$.
This shows that for any $x \in A \cup B, \exists i \in \mathbb{N}_{m+n}$ such that $h(i) = x$.
This proves h is onto.
So, we have shown that there exists a bijection between $A \cup B$ and $\mathbb{N}_{m+n}$, hence $A \cup B$ has $m + n$ elements. $\square$

Theorem 5

Suppose that S and T are sets and that $T \subseteq S$.
(a) If S is a finite set, then T is a finite set.
(b) If T is an infinite set, then S is an infinite set.

Proof. Since (b) is just contrapositive statement of (a). It is enough to prove statement (a).
(a) If $T = \phi$, we already know that T is a finite set. Thus we may suppose that $T \neq \phi$. The proof is by induction on the number of elements in S.
If S has 1 element, then the only nonempty subset T of S must coincide with S, so T is a finite set.
Suppose that every nonempty subset of a set with k elements is finite. Now let S be a set having $k+1$ elements (so there exists a bijection f of $\mathbb{N}_{k+1}$ onto S), and let $T \subseteq S$. If $f(k+1) \notin T$, we can consider T to be a subset of $S_1 := S \setminus \{f(k+1)\}$, which has k elements by previous Theorem. Hence, by the induction hypothesis, T is a finite set.

On the other hand, if $f(k+1) \in T$, then $T_1 := T \setminus \{f(k+1)\}$ is a subset of S_1. Since S_1 has k elements, the induction hypothesis implies that T_1 is a finite set. But this implies that $T = T_1 \cup \{f(k+1)\}$ is also a finite set.

Countable sets

Definition 5

(a) A set S is said to be *denumerable (or countably infinite) if there exists a bijection of N onto S.*

(b) *A set S is said to be countable if it is either finite or denumerable.*

(c) *A set S is said to be uncountable if it is not countable.*

Examples (a) The set $E := \{2n \ : \ n \in \mathbb{N}\}$ of *even* natural numbers is denumerable, since the mapping $f \ ; \mathrm{N} \to E$ defined by $f(n) := 2n$ for $n \in \mathbb{N}$ is a bijection of $\mathbb{N}$ onto E.
Similarly, the set $O := \{2n - 1 \ : \ n \in \mathbb{N}\}$ of *odd* natural numbers is denumerable.
(b) The set $\mathbb{Z}$ of *all* integers is denumerable.

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

(c) The union of two disjoint denumerable sets is denumerable.

Theorem 6

The set $\mathbb{N} \times \mathbb{N}$ is denumerable.

Proof: Recall that $\mathbb{N} \times \mathbb{N} = \{(m, n) : m, n \in \mathbb{N}\}$
We can arrange pairs (m, n) in increasing order of $m + n$. for same $m + n$ we put the pair first for which m is lower.

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), \ldots$$

Theorem 7 (without proof)

Suppose that S and T are sets and that $T \subseteq S$.
(a) If S is a countable set, then T is a countable set.
(b) If T is an uncountable set, then S is an uncountable set.

Theorem 8

The following statements are equivalent;
(a) S is a countable set.
(b) There exists a surjection of $\mathbb{N}$ onto S.
(c) There exists an injection of S into $\mathbb{N}$.

Proof

(a) $\Rightarrow$ (b) If S is finite, there exists a bijection h of some set $\mathbb{N}_n$ onto S and we define H on $\mathbb{N}$ by

$$H(k) := \begin{cases} h(k) & \text{for } k = 1, \ . \ . \ . \ , \ n, \\ h(n) & \text{for } k > n. \end{cases}$$

Then H is a surjection of $\mathbb{N}$ onto S.

If S is denumerable, there exists a bijection H of N onto S, which is also a surjection of N onto S.

(b) $\Rightarrow$ (c) If H is a surjection of $\mathbb{N}$ onto S, we define H_1 ; $S \rightarrow \mathbb{N}$ by letting $H_1(s)$ be the least element in the set $H^{-1}(s) := \{n \in \mathbb{N} : H(n) = s\}$.

To see that H_1 is an injection of S into $\mathbb{N}$, note that if $s, t \in S$ and $n_{\mathrm{t}} := H_1(s) = H_1(t)$, then $s = H(n_l) = t$.

(c) $\Rightarrow$ (a) If H_1 is an injection of S into $\mathbb{N}$, then it is a bijection of S onto $H_1(S) \subseteq \mathbb{N}$.

By above theorem (a), $H_{\mathrm{I}}(S)$ is countable, whence the set S is countable.

Theorem 9

The set $\mathbb{Q}$ of all rational numbers is countable (or denumerable).

proof
Recall that $\mathbb{N} \times \mathbb{N}$ is countable (by Theorem 6), it follows from last Theorem 8(b) that there exists a surjection f of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$.
If $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ is the mapping that sends the ordered pair (m, n) into the rational number having a representation $\frac{m}{n}$, then g is a surjection onto $\mathbb{Q}^+$. Therefore, the composition $g \circ f$ is a surjection of $\mathbb{N}$ onto $\mathbb{Q}^+$, and Theorem implies that $\mathbb{Q}^+$ is a countable set.
Similarly, the set $\mathbb{Q}^-$ of all negative rational numbers is countable. It follows that the set $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$ is countable.
Since $\mathbb{Q}$ contains $\mathbb{N}$, it must be a denumerable set.

Theorem 10

If $\cdot A_n$ is a countable set for each $m \in N$, then the union $A = \cup_{m=1}^{\infty} A_m$ is countable.

Proof. For each $m \in \mathbb{N}$, let φ_m be a surjection of $\mathbb{N}$ onto A_m. We define β ; $\mathbb{N} \times \mathbb{N} \to A$ by

$$\beta(m,\ n) = \varphi_m(n)\ .$$

We claim that β is a surjection. Indeed, if $a \in A$, then there exists a least $m \in \mathbb{N}$ such that $a \in A_m$, whence there exists a least $n \in \mathbb{N}$ such that $a = \varphi_m(n)$. Therefore, $a = \beta(m,\ n)$.
Since $\mathbb{N} \times \mathbb{N}$ is countable, it follows from Theorem 8 that there exists a surjection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ whence $\beta \circ f$ is a surjection of $\mathbb{N}$ onto A.
Now apply Theorem 8 again to conclude that A is countable.

Theorem 11

If A is any set, then there is no surjection of A onto the set $\mathcal{P}(A)$ of all subsets of A.

Proof. Suppose that $\varphi : A \to \mathcal{P}(A)$ is a surjection. Since $\varphi(a)$ is a subset of A, either a belongs to $\varphi(a)$ or it does not belong to this set. We let

$$D = \{a \in A : a \notin \varphi(a)\}.$$

Since D is a subset of A, if φ is a surjection, then $D = \varphi(a_0)$ for some $a_0 \in A$.

We must have either $a_0 \in D$ or $a_0 \notin D$. If $a_0 \in D$, then since $D = \varphi(a_0)$, we must have $a_0 \in \varphi(a_0)$, contrary to the definition of D. Similarly, if $a_0 \notin D$, then $a_0 \notin \varphi(a_0)$ so that $a_0 \in D$, which is also a contradiction. Therefore, φ cannot be a surjection.

Algebraic Properties of $\mathbb{R}$

(A1) $a + b = b + a$ for all a, b in $\mathbb{R}$ (*commutative property of addition*);

(A2) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{R}$ (*associative property of addition*);

(A3) there exists an element 0 in $\mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all a in $\mathbb{R}$ (*existence of a zero element*);

(A4) for each a in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (*existence of negative elements*);

(M1) $a \cdot b = b \cdot a$ for a $1a, b$ in $\mathbb{R}$ (*commutative property of multiplication*);

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{R}$ (*associative property of multiplication*);

(M3) there exists an element 1 in $\mathbb{R}$ *distinct from* 0 such that $1\ .\ a = a$ and $a \cdot 1 = a$ for all a in $\mathbb{R}$ (*existence of a unit element*);

(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ in $\mathbb{R}$ such that $a \cdot (1/a) = 1$ and $(1/a) \cdot a = 1$ (*existence of reciprocals*);

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in $\mathbb{R}$ (*distributive property of multiplication over addition*).

Theorem 12

(a) If z and a are elements in $\mathbb{R}$ with $z + a = a$, then $z = 0$.
(b) If u and $b \neq 0$ are elements in $\mathbb{R}$ with $u \cdot b = b$, then $u = 1$.
(c) If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.

Proof:
(a)

$$\begin{aligned} z &= z + 0 && (\because (A3)) \\ &= z + (a + (-a)) && (\because (A4)) \\ &= (z + a) + (-a) && (\because (A2)) \\ &= a + (-a) && (\because \textit{by hypothesis}) \\ &= 0 && (\because (A4)) \end{aligned}$$

(b)

$$
\begin{aligned}
u &= u \cdot 1 && (\because (M3)) \\
&= u \cdot (b \cdot (\frac{1}{b})) && (\because (M4)) \\
&= (u \cdot b) \cdot (\frac{1}{b}) && (\because (M2)) \\
&= b \cdot (\frac{1}{b}) && (\because by\ \ hypothesis) \\
&= 1 && (\because (M4))
\end{aligned}
$$

(c)

$$
\begin{aligned}
a + a \cdot 0 &= a \cdot 1 + a \cdot 0 && (\because (M3)) \\
&= a \cdot (1 + 0) && (\because (D)) \\
&= a \cdot 1 && (\because (A3)) \\
&= a && (\because (M3)
\end{aligned}
$$

Therefore, by using (a) we can say that $a \cdot 0 = 0$.

Theorem 13

(a) If $a \neq 0$ and b in $\mathbb{R}$ are such that $a \cdot b = 1$, then $b = 1/a$.
(b) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Proof:

(a)

$$
\begin{aligned}
b &= 1 \cdot b & (\because (M3)) \\
&= ((\frac{1}{a}) \cdot a) \cdot b & (\because (M4)) \\
&= (\frac{1}{a}) \cdot (a \cdot b) & (\because (M2)) \\
&= (\frac{1}{a}) \cdot 1 & (\because by \ \ hypotheseis \\
&= \frac{1}{a} & (\because (M3))
\end{aligned}
$$

(b) Let us assume that $a \neq 0$. We have to show $b = 0$.

$$\begin{aligned}
b &= 1 \cdot b && (\because (M3)) \\
&= ((\frac{1}{a}) \cdot a) \cdot b && (\because (M4)\ also\ a \neq 0) \\
&= (\frac{1}{a}) \cdot (a \cdot b) && (\because (M2)) \\
&= (\frac{1}{a}) \cdot 0 && (\because by\ hypotheseis \\
&= 0 && (\because (c)\ of\ theoren\ 1)
\end{aligned}$$

- The operation of subtraction is defined by $a - b = a + (-b)$ for a, b in $\mathbb{R}$.
- Similarly, division is defined for a, b in $\mathbb{R}$ with $b \neq 0$ by $a/b = a \cdot (1/b)$
- We generally write ab instead of $a \cdot b$. Also We will denote $a.a$ as a^2, in general a multiplied n times denoted by a^n.
- We will denote $1/a$ by a^{-1}. Also $1/a^n$ as a^{-n}

Rational and irrational numbers

Recall that,

- $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of Natural numbers.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of Integers.
- $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}$, the set of all rational numbers.
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

Are all the real numbers are rational numbers?
The answer is "NO". There are real numbers other then the rational numbers. we call them *irrational numbers.*
So we can write the set of real numbers $\mathbb{R}$ *as* $\mathbb{Q} \cup \mathbb{Q}^c$

Theorem 14

There does not exist a rational number r such that $r^2 = 2$.

Proof:
We will prove the theorem by method of contradiction. Assume that there exist integers p, q such that $(p/q)^2 = 2$. We can also assume that p and q are positive and have no common factors other than 1. (i.e. $gcd(p, q) = 1$).
Now $(p/q)^2 = 2$ implies $p^2 = 2q^2$ which implies p^2 is even. This implies p is also even. ($\because$ if $p = 2n - 1$ is odd, then $p^2 = (2n-1)^2 = 4n^2 - 4n + 1 = 2(2n^2 - 2n + 1) - 1$ is also odd.). Now since p and q does not have 2 as common factors, q must be odd natural number. (i)
Since p is even, then $p = 2m$ for some $m \in \mathbb{N}$. Hence $p^2 = 2q^2 \Rightarrow 4m^2 = 2q^2 \Rightarrow q^2 = 2m^2$. Therefore q^2 is even, which implies q is even natural number. Which contradicts (i).
This implies our assumption was false. Therefore, there does not exists a rational number r such that $r^2 = 2$.

The Order Properties of $\mathbb{R}$

Definition 6

There is a nonempty subset $\mathbb{R}^+$ (or $\mathbb{P}$) of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:
(i) If a, b belong to $\mathbb{R}^+$, then $a + b$ belongs to $\mathbb{R}^+$.
(ii) If a, b belong to $\mathbb{R}^+$, then ab belongs to $\mathbb{R}^+$.
(iii) If a belongs to $\mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{R}^+, \; a = 0, \; -a \in \mathbb{R}^+.$$

(iii) is usually called *the trichotomy property.*

Definition 7

Let a, b be elements of $\mathbb{R}$.
(a) If $a - b \in \mathbb{R}^+$, then we write $a > b$ or $b < a$.
(b) If $a - b \in \mathbb{R}^+ \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

The trichotomy property implies that for $a, b \in \mathbb{R}$ exactly one of the following holds:

$$a > b, \; a = b, \; a < b$$

Theorem 15

Theorem Let a, b, c be any elements of $\mathbb{R}$.
(a) If $a > b$ and $b > c$, then $a > c$.
(b) If $a > b$, then $a + c > b + c$.
(c) If $a > b$ and $c > 0$, then $ca > cb$.
If $a > b$ and $c < 0$, then $ca < cb$.

proof: (a) If $a - b \in \mathbb{R}^+$ and $b - c \in \mathbb{R}^+$, then definition 6(i) implies that $(a - b) + (b - c) = a - c$ belongs to $\mathbb{R}^+$. Hence $a > c$.
(b) If $a - b \in \mathbb{R}^+$, then $(a + c) - (b + c) = a - b$ is in $\mathbb{R}^+$. Thus $a + c > b + c$.
(c) If $a - b \in \mathbb{R}^+$ and $c \in \mathbb{R}^+$, then $ca - cb = c(a - b)$ is in $\mathbb{P}$ by definition 6(ii). Thus $ca > cb$ when $c > 0$.
On the other hand, if $c < 0$, then $-c \in \mathbb{R}^+$, so that $cbca = (-c)(a - b)$ is in $\mathbb{R}^+$. Thus $cb > ca$ when $c < 0$.

Theorem 16

(a) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.
(b) $1 > 0$.
(c) If $n \in \mathbb{N}$, then $n > 0$.

Proof:
(a) By the Trichotomy Property, if $a \neq 0$, then either $a \in \mathbb{R}^+$ or $-a \in \mathbb{R}^+$. If $a \in \mathbb{R}^+$, then by definition 6(ii), we have $a^2 = a \cdot a \in \mathbb{R}^+$. Also, if $-a \in \mathbb{R}^+$, then $a^2 = (-a)(-a) \in \mathbb{R}^+$. We conclude that if $a \neq 0$, then $a^2 > 0$.
(b) From (a) $1^2 = 1 > 0$.
(c) We use Mathematical Induction. The assertion for $n = 1$ is true by (b). If we suppose the assertion is true for the natural number k, then $k \in \mathbb{R}^+$, and since $1 \in \mathbb{R}^+$, we have $k + 1 \in \mathbb{R}^+$ by definition 6(i). Therefore, the assertion is true for all natural numbers.

Theorem 17

If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

proof:
We'll use method of contradiction. Suppose that $a > 0$. Then if we take $\varepsilon_0 = \frac{1}{2}a$, we have $0 < \varepsilon_0 < a$. Therefore, it is false that $a < \varepsilon$ for every $\varepsilon > 0$. Which contradicts the hypothesis. Hence $a = 0$.

Theorem 18

If $ab > 0$, then either
(i) $a > 0$ and $b > 0$, or (ii) $a < 0$ and $b < 0$.

Proof:
First we note that $ab > 0$ implies that $a \neq 0$ and $b \neq 0$. From the Trichotomy Property, either $a > 0$ or $a < 0$.
If $a > 0$, then $1/a > 0$, and therefore $b = (1/a)(ab) > 0$.
Similarly, if $a < 0$, then $1/a < 0$, so that $b = (1/a)(ab) < 0$.

Examples of Inequalities

(1) Determine the set A of all real numbers x such that $2x + 3 \leq 6$.
We note that we have
$x \in A \;\Leftrightarrow\; 2x + 3 \leq 6 \;\Leftrightarrow\; 2x \leq 3 \;\Leftrightarrow\; x \leq \frac{3}{2}$.
Therefore $A = \{x \in \mathbb{R} \;:\; x \leq \frac{3}{2}\}$.
(2) Determine the set $B := \{x \in \mathbb{R} \;:\; x^2 + x > 2\}$.
We rewrite the inequality so that Theorem 18 can be applied. Note that

$$x \in B \Leftrightarrow x^2 + x - 2 > 0 \Leftrightarrow (x-1)(x+2) > 0.$$

Therefore, we either have
(i) $x - 1 > 0$ and $x + 2 > 0$, or we have (ii) $x - 1 < 0$ and $x + 2 < 0$.
In case (i) we must have both $x > 1$ and $x > -2$ which is satisfied if and only if $x > 1$.
In case (ii) we must have both $x < 1$ and $x < -2$, which is satisfied if and only if $x < -2$.
We conclude that $B = \{x \in \mathbb{R} \;:\; x > 1\} \cup \{x \in \mathbb{R} \;:\; x < -2\}$.

Use of the Order Properties of $\mathbb{R}$ in establishing certain inequalities.

Examples: (1) Let $a \geq 0$ and $b \geq 0$. Then
$a < b \Leftrightarrow a^2 < b^2 \Leftrightarrow \sqrt{a} < \sqrt{b}$
Proof:
If $a = 0$. Then $a < b$ implies $b > 0$. Hence $\sqrt{a} = 0 < \sqrt{b}$.
We consider the case where $a > 0$ and $b > 0$.
It follows from definition 6(i) that $a + b > 0$. Since
$b^2 - a^2 = (b - a)(b + a)$, it follows from Theorem 15(c) that $b - a > 0$ implies that $b^2 - a^2 > 0$. Also, it follows from Theorem 18 that $b^2 - a^2 > 0$ implies that $b - a > 0$.
If $a > 0$ and $b > 0$, then $\sqrt{a} > 0$ and $\sqrt{b} > 0$. Since $a = (\sqrt{a})^2$ and $b = (\sqrt{b})^2$, the second implication is a consequence of the first one when a and b are replaced by $\sqrt{a}$ and $\sqrt{b}$, respectively.

Example: (2) (b) If a and b are positive real numbers, then their *arithmetic mean is* $\frac{1}{2}(a+b)$ *and their geometric mean is* $\sqrt{ab}$. *The Arithmetic-Geometric Mean Inequality for* a, b *is*

$$\sqrt{ab} \leq \frac{1}{2}(a+b)$$

with equality occurring if and only if $a = b$.

Proof:

Note that if $a > 0, b > 0$, *and* $a \neq b$, *then* $\sqrt{a} > 0, \sqrt{b} > 0$, *and* $\sqrt{a} \neq \sqrt{b}$.

Therefore it follows from Theorem 16(a) that $(\sqrt{a} - \sqrt{b})^2 > 0$. *Expanding this square, we obtain*

$a - 2\sqrt{ab} + b > 0 \Rightarrow \sqrt{ab} < \frac{1}{2}(a+b)$.

If $a = b$, *then* $\sqrt{ab} = \sqrt{a \cdot a} = \sqrt{a^2} = a = \frac{1}{2}(a+a) = \frac{1}{2}(a+b)$.

On the other hand, suppose that $a > 0, b > 0$ and that $\sqrt{ab} = \frac{1}{2}(a+b)$. Then, squaring both sides and multiplying by 4, we obtain

$$4ab = (a+b)^2 = a^2 + 2ab + b^2 \Rightarrow 0 = a^2 - 2ab + b^2 = (a-b)^2$$

But this equality implies that $a = b$. (Why?) Thus, equality in (2) implies that $a = b$.

Remark: The general Arithmetic-Geometric Mean Inequality for the positive real numbers $a_1, a_2, \ldots, a_n$ is

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality occurring if and only if $a_1 = a_2 = \cdots = a_n$.

(3) *Bernoulli's Inequality.*

If $x > -1$, *then* $(1+x)^n \geq 1 + nx \quad$ *for all* $\quad n \in \mathbb{N}$ *(*)*

We'll use Mathematical Induction. The case $n = 1$ *yields equality, so the statement is true in this case. Next, we assume the inequality (*) is true for* $k \in \mathbb{N}$ *and will deduce it for* $k + 1$. *Indeed, the assumptions that* $(1+x)^k \geq 1 + kx$ *and that* $1 + x > 0$ *imply that*

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\ &\geq (1+kx) \cdot (1+x) = 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x \end{aligned}$$

Thus, inequality (*) holds for $n = k + 1$. *Therefore, (*) holds for all* $n \in \mathbb{N}$

Absolute Value and the Real Line

Definition 8

The absolute value of a real number a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

For example, $|5| = 5$ and $|-8| = 8$.
We see from the definition that $|a| \geq 0$ for all $a \in \mathbb{R}$, and that $|a| = 0$ if and only if $a = 0$.
Also $|-a| = |a|$ for all $a \in \mathbb{R}$.

Theorem 19

(a) $|ab| = |a||b|$ *for all* $a, b \in \mathbb{R}$
(b) $|a|^2 = a^2$ *for all* $a \in \mathbb{R}$
(c) If $c \geq 0$, *then* $|a| \leq c$ *if and only if* $-c \leq a \leq c$
(d) $-|a| \leq a \leq |a|$ *for all* $a \in \mathbb{R}$

Proof. (a) If either a or b is 0 , then both sides are equal to 0 . There are four other cases to consider.
If $a > 0, b > 0$, then $ab > 0$, so that $|ab| = ab = |a||b|$.
If $a > 0, b < 0$, then $ab < 0$, so that $|ab| = -ab = a(-b) = |a||b|$.
The remaining cases can be proved similarly.
(b) since $a^2 \geq 0$, we have $a^2 = \left|a^2\right| = |aa| = |a||a| = |a|^2$
(c) If $|a| \leq c$, then we have both $a \leq c$ and $-a \leq c$, which is equivalent to $-c \leq a \leq c$ Conversely, if $-c \leq a \leq c$, then we have both $a \leq c$ and $-a \leq c$, so that $|a| \leq c$
(d) Take $c = |a|$ in part (c).

Theorem 20

Triangle Inequality If $a, b \in \mathbb{R}$, *then* $|a+b| \leq |a| + |b|$

Proof. From (d) part of last theorem, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. On adding these inequalities, we obtain

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

Hence, by (c) part of last theorem, we have $|a+b| \leq |a| + |b|$ It can be shown that equality occurs in the Triangle Inequality if and only if $ab > 0$ which is equivalent to saying that a and b have the same sign. There are many useful variations of the Triangle Inequality. Here are two.

Theorem 21 (Corollary)

If $a, b \in \mathbb{R}$, *then*
(a) $||a| - |b|| \leq |a - b|$
(b) $|a - b| \leq |a| + |b|$

Proof. (a) We write $a = a - b + b$ and then apply the Triangle Inequality to get $|a| = |(a - b) + b| \leq |a - b| + |b|$.
Now subtract $|b|$ to get $|a| - |b| \leq |a - b|$.
Similarly, from $|b| = |b - a + a| \leq |b - a| + |a|$, we obtain
$-|a - b| = -|b - a| \leq |a| - |b|$.
If we combine these two inequalities, using (c) part of Theorem 20, we get the inequality in (a)
(b) By replacing b in the Triangle Inequality by $-b$ we get
$|a - b| \leq |a| + |-b|$.
since $|-b| = |b|$ we obtain the inequality in (b).

Definition 9

The distance between elements a and b in $\mathbb{R}$ is $|a-b|$.

Definition 10

Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε -neighborhood of a is the set

$$N_c(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$$

Theorem 22

Let $a \in \mathbb{R}$. If x belongs to the neighborhood $N_c(a)$ for every $\varepsilon > 0$, then $x = a$

Proof: If a particular x satisfies $|x-a| < \varepsilon$ for every $\varepsilon > 0$, then it follows from 17 that $|x-a| = 0$, and hence $x = a$

Definition 11

Let S be a nonempty subset of $\mathbb{R}$

(a) The set S is said to be bounded above if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. Each such number u is called an upper bound of S.

(b) The set S is said to be bounded below if there exists a number $w \in \mathbb{R}$ such that $w \leq s$ for all $s \in S$. Each such number w is called a lower bound of S.

(c) A set is said to be bounded if it is both bounded above and bounded below. A set is said to be unbounded if it is not bounded.

For example, the set $S := \{x \in \mathbb{R} : x < 2\}$ is bounded above; the number 2 and any number larger than 2 is an upper bound of S. This set has no lower bounds, so that the set is not bounded below. Thus it is unbounded (even though it is bounded above)

If a set has one upper bound, then it has infinitely many upper bounds, because if u is an upper bound of S, then the numbers $u + 1, u + 2, \ldots$ are also upper bounds of S. (A similar observation is valid for lower bounds. In the set of upper bounds of S and the set of lower bounds of S, we single out their least and greatest elements, respectively, for special attention in the following definition.

Definition 12

Let S be a nonempty subset of $\mathbb{R}$

(a) If S is bounded above, then a number u is said to be a supremum (or a least upper bound) of S if it satisfies the conditions:

(1) u is an upper bound of S, and

(2) if v is any upper bound of S, then $u \leq v$

(b) If S is bounded below, then a number w is said to be an infimum (or a greatest lower bound) of S if it satisfies the conditions:

(1') w is a lower bound of S, and

(2') if t is any lower bound of S, then $t \leq w$

There can be only one supremum of a given subset S of $\mathbb{R}$.
For, suppose that u_1 and u_2 are both supremum of S. If $u_1 < u_2$, then the hypothesis that u_2 is a supremum implies that u_1 cannot be an upper bound of S. Similarly, we see that $u_2 < u_1$ is not possible. Therefore, we must have $u_1 = u_2$.
A similar argument can be given to show that the infimum of a set is uniquely determined.
If the supremum or the infimum of a set S exists, we will denote them by $sup\ S$ and $inf\ S$ or $sup\ (S)$ and $inf\ (S)$

Lemma *A number u is the supremum of a nonempty subset S of $\mathbb{R}$ if and only if u satisfies the conditions:*
(1) $s \leq u$ for all $s \in S$
(2) if $v < u$, then there exists $s' \in S$ such that $v < s'$
Lemma *An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_c \in S$ such that $u - \varepsilon < s_c$.*
Proof. If u is an upper bound of S that satisfies the stated condition and if $v < u$, then we put $\varepsilon := u - v$. Then $\varepsilon > 0$, so there exists $s_c \in S$ such that $v = u - \varepsilon < s_t$. Therefore, v is not an upper bound of S, and we conclude that $u = \sup S$
Conversely, suppose that $u = \sup S$ and let $\varepsilon > 0$. since $u - \varepsilon < u$, then $u - \varepsilon$ is not an upper bound of S. Therefore, some element s_ε of S must be greater than $u - \varepsilon$; that is, $u - \varepsilon < s_c$)

Examples

(a) If a nonempty set S_1 has a finite number of elements, then it can be shown that S_1 has a largest element u and a least element w. Then $u = \sup S_1$ and $w = \inf S_1$ and they are both members of S_1.

(b) The set $S_2 := \{x : 0 \leq x \leq 1\}$ clearly has 1 for an upper bound. We prove that 1 is its supremum as follows. If $v < 1$, there exists an element $s' \in S_2$ such that $v < s'$. (Name one such element s'.) Therefore v is not an upper bound of S_2 and, since v is an arbitrary number $v < 1$, we conclude that $\sup S_2 = 1$. It is similarly shown that $\inf S_2 = 0$. Note that both the supremum and the infimum of S_2 are contained in S_2

(c) The set $S_3 := \{x : 0 < x < 1\}$ clearly has 1 for an upper bound. Using the same argument as given in (b), we see that $\sup S_3 = 1$. In this case, the set S_3 does not contain its supremum. Similarly, $\inf S_3 = 0$ is not contained in S_3

The Completeness Property of $\mathbb{R}$: Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

Theorem 23 (Archimedean Property)

If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \leq n_x$

Proof. If the statement is false, then $n \leq x$ for all $n \in \mathbb{N}$; therefore, x is an upper bound of $\mathbb{N}$.
Therefore, by the Completeness Property, the nonempty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$.
Subtracting 1 from u gives a number $u - 1$, which is smaller than the supremum u of $\mathbb{N}$.
Therefore $u - 1$ is not an upper bound of $\mathbb{N}$, so there exists $m \in \mathbb{N}$ with $u - 1 < m$.
Adding 1 gives $u < m + 1$.
And since $m + 1 \in \mathbb{N}$, this inequality contradicts the fact that u is an upper bound of $\mathbb{N}$.
Which proves that the given statement is correct.

Corollary 1

Corollary If $S := \{1/n : n \in \mathbb{N}\}$, *then inf* $S = 0$

Proof. since $S \neq \emptyset$ is bounded below by 0 , it has an infimum and we let $w = \inf S$. Since S is bounded below by 0, $inf\ S = w \geq 0$.
For any $\varepsilon > 0$, the Archimedean Property implies that there exists $n \in \mathbb{N}$ such that $1/\varepsilon < n$, which implies $1/n < \varepsilon$.
Therefore we have

$$0 \leq w \leq 1/n < \varepsilon$$

But since $\varepsilon > 0$ is arbitrary, it follows from known Theorem that $w = 0$.

Corollary 2

If $t > 0$, *there exists* $n_t \in \mathbb{N}$ *such that* $0 < 1/n_t < t$

Proof. since $\inf\{1/n : n \in \mathbb{N}\} = 0$ and $t > 0$, then t is not a lower bound for the set $\{1/n : n \in \mathbb{N}\}$. Thus there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$

Corollary 3

If $y > 0$, *there exists* $n_y \in \mathbb{N}$ *such that* $n_y - 1 \leq y \leq n_y$

Proof. Using the Archimedean Property we can say that the subset $E_y = \{m \in \mathbb{N} : y < m\}$ of $\mathbb{N}$ is not empty.
By the Well-Ordering Property E_y has a least element, which we denote by n_y.
Then $n_y - 1$ does not belong to E_y, and hence we have $n_y - 1 \leq y < n_y$.

Theorem 24

There exists a real number x such that $x^2 = 2$.

Theorem 25 (The Density Theorem)

If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$

Proof. Without loss of generality assume that $x > 0$.
Since $y - x > 0$, it follows from Corollary 2 that there exists $n \in \mathbb{N}$ such that $1/n < y - x$. Therefore, we have $nx + 1 < ny$.
If we apply Corollary 3 to $nx > 0$, we get $m \in \mathbb{N}$ with $m - 1 \leq nx < m$.
Therefore, $m \leq nx + 1 < ny$, and $nx < m < ny$.
Hence, the rational number $r = m/n$ satisfies $x < r < y$.

Corollary 4

If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$

Proof. If we apply the Density Theorem to the real numbers $x/\sqrt{2}$ and $y/\sqrt{2}$, we obtain a rational number $r \neq 0$, such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

Then $z = r\sqrt{2}$ is irrational and satisfies $x < z < y$.