SHRI GOVIND GURU UNIVERSITY
B.Sc.Sem-5 Material
BSC0C506C:Mathematics(Theory)
Linear Algebra-II

Unit-I

Unit-I: Composition of Linear Maps,The Space $L(U,V)$,The Operator Equation,Linear Functional,Dual Space,Dual of Dual,Dual Basis Existence Theorem,Annihilators, bilinear forms.

1 Linear Transformation

Definition 1.1 *Let U and V are two vector space then a mapping $T:U\to V$ is called a Linear Transformation if it satisfies the following condition:*

1. $\forall\ \overline{x},\overline{y}\in U,\ \ T(\overline{x}+\overline{y})=T(\overline{x})+T(\overline{y})$
2. $\forall\ \overline{x}\in U,\ \alpha\in\mathbb{R},\ \ T(\alpha\overline{x})=\alpha T(\overline{x})$

Definition 1.2 *Let $T:U\to V$ and $S:U\to V$ be two Linear Transformation then the sum of T and S is denoted by $T+S$ and defined as* $T+S:U\to V$

$$(T+S)(\overline{x})=T(\overline{x})+S(\overline{x}),\ \forall\,\overline{x}\in U$$

Example 1 $T:\mathbb{R}^2\to\mathbb{R}^3,\ T(\overline{x})=(x+y,x-y,0),\ S:\mathbb{R}^2\to\mathbb{R}^3,\ S(\overline{x})=(x-y,x+y,2x)$, *then find $(T+S)$.*

Solution:

$$\begin{aligned}(T+S)(\overline{x})&=T(\overline{x})+S(\overline{x})\\&=(x+y,x-y,0)+(x-y,x+y,2x)\\&=(x+y+x-y,x-y+x+y,0+2x)\\&=(2x,2x,2x)\end{aligned}$$

Definition 1.3 *Let $T:U\to V$ be a Linear Transformation and let α be a scalar then the scalar multiplication of a linear transformation T by α denoted by αT and defined as $\alpha T:U\to V$*

$$(\alpha T)(\overline{x})=\alpha T(\overline{x}),\ \ \forall\,\overline{x}\in U$$

Definition 1.4 *The set f all Linear Transformation from U to V is denoted by $L(U,V)$.*

$$L(U,V)=\{T/T:U\to V\ is\ a\ linear\ transformation\}$$

Definition 1.5 *Let $T:U\to V$ be a linear transformation and let $S:V\to W$ be a linear transformation then, the composition of S and T is denoted by SoT and defined as $SoT:U\to W$.*

$$SoT(\overline{x})=S(T(\overline{x})),\ \ \forall\,\overline{x}\in U$$

Theorem 1 *Prove that the sum of two linear transformation is also linear transformation.*

OR

If $T,S\in L(U,V)$ then prove that $S+T\in L(U,V)$.

Proof: *Here $T,S\in L(U,V)$ i.e. $T:U\to V$ and $S:U\to V$ are linear transformation.And we have to prove $S+T:U\to V$ is also linear transformation.*

(i) Let $\overline{x}, \overline{y} \in U$ *to prove that* $(S+T)(\overline{x}+\overline{y}) = (S+T)(\overline{x}) + (S+T)(\overline{y})$

$$\begin{aligned}
(S+T)(\overline{x}+\overline{y}) &= S(\overline{x}+\overline{y}) + T(\overline{x}+\overline{y}) && (\because By\ \ Definition(1.2)) \\
&= (S(\overline{x}) + S(\overline{y})) + (T(\overline{x}) + T(\overline{y})) && (\because\ S, T\ areL.T.) \\
&= S(\overline{x}) + T(\overline{x}) + S(\overline{y}) + T(\overline{y}) \\
&= (S+T)(\overline{x}) + (S+T)(\overline{y}) && (\because By\ \ Definition(1.2))
\end{aligned}$$

(ii) Let $\alpha \in \mathbb{R}$ *and let* $x \in U$ *to prove that* $(S+T)(\alpha\overline{x}) = \alpha(S+T)(\overline{x})$.

$$\begin{aligned}
(S+T)(\alpha\overline{x}) &= S(\alpha\overline{x}) + T(\alpha\overline{x}) && (\because By\ \ Definition(1.2)) \\
&= \alpha S(\overline{x}) + \alpha T(\overline{x}) && (\because\ S, T\ areL.T.) \\
&= \alpha(S(\overline{x}) + T(\overline{x})) \\
&= \alpha(S+T)(\overline{x})
\end{aligned}$$

So from (i) and (ii) $S+T : U \to V$ *is also linear transformation.*

Theorem 2 *If* $T \in L(U,V)$ *and* $\alpha \in \mathbb{R}$ *then prove that* $\alpha T \in L(U,V)$. ***Proof:*** *Here* $T : U \to V$ *is a linear transformation and* α *be a scalar to prove that* $\alpha T : U \to V$ *is linear transformation.*

(i) Let $x, y \in U$ *to prove that* $(\alpha T)(\overline{x}+\overline{y}) = (\alpha T)(\overline{x}) + (\alpha T)(\overline{y})$

$$\begin{aligned}
(\alpha T)(\overline{x}+\overline{y}) &= \alpha(T(\overline{x}+\overline{y})) && (\because By\ \ Definition(1.3)) \\
&= \alpha(T(\overline{x}) + T(\overline{y})) && (\because T\ is\ L.T.) \\
&= \alpha T(\overline{x}) + \alpha T(\overline{y}) \\
&= (\alpha T)(\overline{x}) + (\alpha T)(\overline{y})
\end{aligned}$$

(ii) Let $x \in U$ *and let* β *be a scalar* $\beta \in \mathbb{R}$ *to prove that* $(\alpha T)(\beta\overline{x}) = \beta((\alpha T)(\overline{x}))$

$$\begin{aligned}
(\alpha T)(\beta\overline{x}) &= \alpha(T(\beta(\overline{x}))) && (\because By\ \ Definition(1.3)) \\
&= \alpha(\beta(T(\overline{x}))) && (\because T\ is\ L.T.) \\
&= (\alpha\beta)T(\overline{x}) \\
&= (\beta\alpha)T(\overline{x}) \\
&= \beta((\alpha T)(\overline{x})) && (\because By\ \ Definition(1.3))
\end{aligned}$$

From (i) and (ii) $\alpha T : U \to V$ *is a linear transformation.*

Theorem 3 *The composition of two linear transformation is also a linear transformation.*

OR

If $T \in L(U,V)$ *and* $S \in L(V,W)$,*then prove that* $SoT \in L(V,W)$.

Proof: *Here* $T \in L(U,V)$, *so* $T : U \to V$ *is a linear transformation and* $S \in L(V,W)$ *so* $S : V \to W$ *is a linear transformation.*
And we have to prove that $SoT : U \to W$ *is also linear transformation.*

(i) Let $\overline{x}, \overline{y} \in U$ to prove that $(SoT)(\overline{x}+\overline{y}) = (SoT)(\overline{x}) + (SoT)(\overline{y})$.

$$\begin{aligned}
(SoT)(\overline{x}+\overline{y}) &= S(T(\overline{x}+\overline{y})) && (\because By\ \ Definition(1.5))\\
&= S(T(\overline{x})+T(\overline{y})) && (\because T\ is\ L.T.)\\
&= S(T(\overline{x}))+S(T(\overline{y})) && (\because S\ is\ L.T.)\\
&= (SoT)(\overline{x})+(SoT)(\overline{y}) && (\because By\ \ Definition(1.5))
\end{aligned}$$

(ii) Let $\overline{x} \in U$ and let α be a scalar to prove that $(SoT)(\alpha\overline{x}) = \alpha((SoT)(\overline{x}))$.

$$\begin{aligned}
(SoT)(\alpha\overline{x}) &= S(T(\alpha\overline{x})) && (\because By\ \ Definition(1.5))\\
&= S(\alpha T(\overline{x}))) && (\because T\ is\ L.T.)\\
&= \alpha(S(T(\overline{x}))) && (\because S\ is\ L.T.)\\
&= \alpha((SoT)(\overline{x})) && (\because By\ \ Definition(1.5))
\end{aligned}$$

So from (i) and (ii) $SoT : U \to W$ is a linear transformation.

Example 2 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x-y, x+y)$, $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(x, y) = (x+y, x-y)$ then find SoT and ToS.*
Solution: *Let $(x, y) \in \mathbb{R}^2$*

$$\begin{aligned}
(SoT)(x,y) &= S(T(x,y))\\
&= S(x-y, x+y)\\
&= (x-y+x+y, x-y-x-y)\\
&= (2x, -2y)\\
(ToS)(x,y) &= T(S(x,y))\\
&= T(x+y, x-y)\\
&= (x+y-x+y, x+y+x-y)\\
&= (2y, 2x)
\end{aligned}$$

2 Linear functional and Dual Space

Definition 2.1 *Let V be a real vector space then a mapping $f : V \to \mathbb{R}$ is said be a linear functional if it satisfies the following conditions:*

1. $f(x+y) = f(x) + f(y), \quad \forall\, x, y \in V$
2. $f(\alpha x) = \alpha f(x), \qquad \forall\ x \in V\ \ and\ \alpha \in \mathbb{R}$

Note: The set of all linear functional from V to $\mathbb{R}$ is denoted by $L(V, \mathbb{R})$ or V^*.

$$L(V, \mathbb{R}) = V^* = \{f/f : V \to \mathbb{R}\ is\ a\ linear\ functional\}$$

Definition 2.2 *Let V^* be the set of all linear functional from V to $\mathbb{R}$,where V is a vector space. for $f, g \in V^*$ and $\alpha \in \mathbb{R}$,*

1. $(f+g)(x) = f(x) + g(x), \quad \forall\, x \in V$
2. $(\alpha f)(x) = \alpha(f(x)) \quad \forall\, x \in V, \ \ \alpha \in \mathbb{R}$

*under this operation V^* is a vector space and this vector space V^* is called a **Dual space** of a vector space V.*

Theorem 4 *State and Prove Dual Basis existence theorem.*

Statement: *Let* $B = \{v_1, v_2, \ldots, v_n\}$ *be a basis for a vector space* V*.let* V^* *be a dual space of* V*, suppose* $f_1, f_2, \ldots f_n \in V^*$ *such that*

$$\begin{aligned} f_i(v_j) &= 1 \qquad i = j \\ &= 0 \qquad i \neq j \quad i, j = 1, 2, \ldots n \end{aligned}$$

Then prove that $B^* = \{f_1, f_2, \ldots f_n\}$ *is a basis for* V^*.

Proof: *Here* $B = \{v_1, v_2, \ldots, v_n\}$ *be a basis for a vector space* V *and* V^* *be a dual space of* V*, and* $f_1, f_2, \ldots f_n \in V^*$ *such that*

$$\begin{aligned} f_i(v_j) &= 1 \qquad i = j \\ &= 0 \qquad i \neq j \quad i, j = 1, 2, \ldots n \end{aligned} \tag{1}$$

we have tp prove $B^* = \{f_1, f_2, \ldots f_n\}$ *is a basis for* V^*.

(i) First we shall prove that B^* *is Linearly Independent*

Consider,

$$\begin{aligned} \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n &= \overline{0} \qquad where \;\; \alpha_i \in \mathbb{R}, \;\; i = 1, 2, \ldots, n. \\ (\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n)(v_1) &= \overline{0}(v_1) \\ (\alpha_1 f_1)(v_1) + (\alpha_2 f_2)(v_1) + \ldots + (\alpha_n f_n)(v_1) &= 0 \\ \alpha_1 (f_1)(v_1) + \alpha_2 (f_2)(v_1) + \ldots + \alpha_n (f_n)(v_1) &= 0 \qquad (\because By \;\; Definition (1.3)) \\ \alpha_1(1) + \alpha_2(0) + \ldots + \alpha_n(0) &= 0 \qquad (By \;\; Equation \; (1)) \\ \alpha_1(1) &= 0 \\ \alpha_1 &= 0 \end{aligned}$$

Similarly, we can prove $\alpha_2 = 0, \alpha_3 = 0, \ldots, \alpha_n = 0$.
so $B^* = \{f_1, f_2, \ldots f_n\}$ *is Linearly Independent.*

(ii) Now we have to prove that $[B^*] = V^*$.

we know that $[B^*] \subseteq V^*$.
so only to prove $V^* \subseteq [B^*]$
take $f \in V^*$, *so* $f : V \to \mathbb{R}$ *is a linear functional.*
Suppose,

$$\begin{aligned} f(v_1) &= \alpha_1 \\ f(v_2) &= \alpha_2 \\ &\vdots \\ f(v_n) &= \alpha_n, \quad where \;\; \alpha_i \in \mathbb{R}, \;\; i = 1, 2, \ldots, n. \end{aligned}$$

Let us define a function $\phi : V \to \mathbb{R}$ *such that*

$$\phi = \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n \tag{2}$$

Now,

$$\begin{aligned}
\phi(v_1) &= (\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n)(v_1) \\
&= (\alpha_1 f_1)(v_1) + (\alpha_2 f_2)(v_1) + \ldots + (\alpha_n f_n)(v_1) \\
&= \alpha_1 (f_1)(v_1) + \alpha_2 (f_2)(v_1) + \ldots + \alpha_n (f_n)(v_1) && (\because By\ \ Definition (1.3)) \\
&= \alpha_1(1) + \alpha_2(0) + \ldots + \alpha_n(0) && (By\ \ Equation\ (1)) \\
\phi(v_1) &= \alpha_1
\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}
\phi(v_2) &= \alpha_2 \\
\phi(v_3) &= \alpha_3 \\
&\vdots \\
\phi(v_n) &= \alpha_n \\
So,\ \ \phi(v_i) &= \alpha_i, \quad where\ i = 1, 2, \ldots n
\end{aligned}$$

also here

$$\begin{aligned}
f(v_i) &= \alpha_i \\
\phi(v_i) &= f(v_i) \\
\phi &= f
\end{aligned}$$

so by equation (2)

$$\begin{aligned}
f &= \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n \\
f &\in [B^*] \\
V^* &\subseteq [B^*]
\end{aligned}$$

so,

$$[B^*] = V^*$$

so from (i) and (ii) $B^* = \{f_1, f_2, \ldots f_n\}$ *is a basis for* V^*.

Definition 2.3 *Let* V *be a vector space and* V^* *be a dual space of a vector space* V*.Let* $\dim V = n$ *then* $\dim V^* = n$ *and basis* $B^* = \{f_1, f_2, \ldots f_n\}$ *of* V^* *corresponding to a basis* $B = \{v_1, v_2, \ldots, v_n\}$ *of a vector space* V *is called a dual basis for a vector space* V.

Example 3 *Find the dual basis corresponding to a basis* $\{(2,1),(3,1)\}$ *of* $\mathbb{R}^2$.
Solution: *Here* $\mathbb{R}^2$ *is a vector space*
$\therefore \dim \mathbb{R}^2 = 2$
Let $(\mathbb{R}^2)^*$ *be a dual space of* $\mathbb{R}^2$.
$\therefore \dim(\mathbb{R}^2)^* = 2$
Also here $B = \{(2,1),(3,1)\}$ *is a basis for* $\mathbb{R}^2$.
let $v_1 = (2,1)$ *and* $v_2 = (3,1)$
to find $B^* = \{f_1, f_2\}$ *a dual basis for* $\mathbb{R}^2$.
Define function $f_1 : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$f_1(x,y) = ax + by, \quad a, b \in \mathbb{R}$$

$$\begin{aligned} f_1(x,y) &= ax+by \\ f_1(v_1) &= ax+by \\ f_1(2,1) &= 2a+b \\ 1 &= 2a+b \\ 2a+b=1 & \end{aligned} \tag{3}$$

$$\begin{aligned} f_1(x,y) &= ax+by \\ f_1(v_2) &= ax+by \\ f_1(3,1) &= 3a+b \\ 0 &= 3a+b \\ 3a+b=0 & \end{aligned} \tag{4}$$

Solve equation (3) and (4) we get $a=-1$.
Substitute $a=-1$ *in equation (3) we get* $b=3$.
So,we get

$$f_1(x,y) = -x+3y$$

Now, we define function $f_2 : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$f_2(x,y) = cx+dy, \quad c,d \in \mathbb{R}$$

$$\begin{aligned} f_2(x,y) &= cx+dy \\ f_2(v_1) &= cx+dy \\ f_2(2,1) &= 2c+d \\ 0 &= 2c+d \\ 2c+d=0 & \end{aligned} \tag{5}$$

$$\begin{aligned} f_2(x,y) &= cx+dy \\ f_2(v_2) &= cx+dy \\ f_2(3,1) &= 3c+d \\ 1 &= 3c+d \\ 3c+d=1 & \end{aligned} \tag{6}$$

Solve equation (5) and (6) we get $c=1$.
Substitute $c=1$ *in equation (5) we get* $d=-2$.
So,we get

$$f_2(x,y) = x-2y$$

Thus $B^* = \{f_1, f_2\}$ *is a dual basis for* $\mathbb{R}^2$ *.*
where,

$$\begin{aligned} f_1(x,y) &= -x+3y \\ f_2(x,y) &= x-2y \end{aligned}$$

Example 4 *Find the dual basis corresponding to a basis* $\{(1,0,1),(1,1,0),(0,1,1)\}$ *of* $\mathbb{R}^3$.
Solution: *Here* $\mathbb{R}^3$ *is a vector space*

$\therefore \dim \mathbb{R}^3 = 3$
Let $(\mathbb{R}^3)^*$ *be a dual space of* $\mathbb{R}^3$.
$\therefore \dim(\mathbb{R}^3)^* = 3$
Also here $B = \{(1,0,1),(1,1,0),(0,1,1)\}$ *is a basis for* $\mathbb{R}^3$.
let $v_1 = (1,0,1)$, $v_2 = (1,1,0)$ *and* $v_3 = (0,1,1)$
to find $B^* = \{f_1, f_2, f_3\}$ *a dual basis for* $\mathbb{R}^3$
Let $v_1 = (1,0,1), v_2 = (1,1,0)\ \ and\ \ v_3 = (0,1,1)$
to find $B^* = \{f_1, f_2, f_3\}$ *a dual basis for* $\mathbb{R}^3$.
Define function $f_1 : \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$f_1(x,y,z) = ax + by + cz, \quad a,b,c \in \mathbb{R}$$

$$\begin{aligned} f_1(x,y,z) &= ax + by + cz \\ f_1(v_1) &= ax + by + cz \\ f_1(1,0,1) &= a + c \\ 1 &= a + c \\ a + c &= 1 \end{aligned} \tag{7}$$

$$\begin{aligned} f_1(x,y,z) &= ax + by + cz \\ f_1(v_2) &= ax + by + cz \\ f_1(1,1,0) &= a + b \\ 0 &= a + b \\ a + b &= 0 \\ a &= -b \end{aligned} \tag{8}$$

$$\begin{aligned} f_1(x,y,z) &= ax + by + cz \\ f_1(v_3) &= ax + by + cz \\ f_1(0,1,1) &= b + c \\ 0 &= b + c \\ b + c &= 0 \end{aligned} \tag{9}$$

from equation (8) $a = -b$ *in equation (7) we get*

$$b - c = -1 \tag{10}$$

solve equation (9) and (10) we get $b = \dfrac{-1}{2}$.
Substitute $b = \dfrac{-1}{2}$ *in equation (8) we get* $a = \dfrac{1}{2}$.
from equation (9) we get $c = \dfrac{1}{2}$.
Thus we get,

$$\begin{aligned} f_1(x,y,z) &= \frac{1}{2}x - \frac{1}{2}y + \frac{1}{2}z \\ f_1(x,y,z) &= \frac{1}{2}(x - y + z) \end{aligned}$$

Similarly we define function $f_2 : \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$f_2(x, y, z) = ax + by + cz, \quad a, b, c \in \mathbb{R}$$

$$\begin{aligned} f_2(x, y, z) &= ax + by + cz \\ f_2(v_1) &= ax + by + cz \\ f_2(1, 0, 1) &= a + c \\ 0 &= a + c \\ a + c &= 0 \\ a &= -c \end{aligned} \tag{11}$$

$$\begin{aligned} f_2(x, y, z) &= ax + by + cz \\ f_2(v_2) &= ax + by + cz \\ f_2(1, 1, 0) &= a + b \\ 1 &= a + b \\ a + b &= 1 \end{aligned} \tag{12}$$

$$\begin{aligned} f_2(x, y, z) &= ax + by + cz \\ f_2(v_3) &= ax + by + cz \\ f_2(0, 1, 1) &= b + c \\ 0 &= b + c \\ b + c &= 0 \end{aligned} \tag{13}$$

from equation (11) $a = -c$ *in equation (12) we get*

$$b - c = 1 \tag{14}$$

solve equation (13) and (14) we get $b = \dfrac{1}{2}$.
Substitute $b = \dfrac{1}{2}$ *in equation (14) we get* $c = \dfrac{-1}{2}$.
from equation (11) we get $a = \dfrac{1}{2}$.
Thus we get,

$$\begin{aligned} f_2(x, y, z) &= \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z \\ f_2(x, y, z) &= \frac{1}{2}(x + y - z) \end{aligned}$$

Now we define function $f_3 : \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$f_3(x, y, z) = ax + by + cz, \quad a, b, c \in \mathbb{R}$$

$$\begin{aligned} f_3(x, y, z) &= ax + by + cz \\ f_3(v_1) &= ax + by + cz \\ f_3(1, 0, 1) &= a + c \\ 0 &= a + c \\ a + c &= 0 \\ a &= -c \end{aligned} \tag{15}$$

$$\begin{aligned} f_3(x,y,z) &= ax+by+cz \\ f_3(v_2) &= ax+by+cz \\ f_3(1,1,0) &= a+b \\ 0 &= a+b \\ a+b &= 0 \end{aligned} \tag{16}$$

$$\begin{aligned} f_3(x,y,z) &= ax+by+cz \\ f_3(v_3) &= ax+by+cz \\ f_3(0,1,1) &= b+c \\ 1 &= b+c \\ b+c &= 1 \end{aligned} \tag{17}$$

from equation (15) $a = -c$ in equation (16) we get

$$b - c = 0 \tag{18}$$

solve equation (17) and (18) we get $b = \dfrac{1}{2}$.
Substitute $b = \dfrac{1}{2}$ in equation (17) we get $c = \dfrac{1}{2}$.
from equation (15) we get $a = \dfrac{-1}{2}$.
Thus we get,

$$\begin{aligned} f_3(x,y,z) &= \frac{-1}{2}x + \frac{1}{2}y + \frac{1}{2}z \\ f_3(x,y,z) &= \frac{1}{2}(-x+y+z) \end{aligned}$$

Thus $B^* = \{f_1, f_2, f_3\}$ is a dual basis for $\mathbb{R}^3$.
where,

$$\begin{aligned} f_1(x,y,z) &= \frac{1}{2}(x-y+z) \\ f_2(x,y,z) &= \frac{1}{2}(x+y-z) \\ f_3(x,y,z) &= \frac{1}{2}(-x+y+z) \end{aligned}$$

Theorem 5 *Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and $\{f_1, f_2, \ldots, f_n\}$ be a basis for V^* then prove that for any $v \in V$*

$$v = f_1(v)v_1 + f_2(v)v_2 + \ldots + f_n(v)v_n$$

and for any $f \in V^*$

$$f = f(v_1)f_1 + f(v_2)f_2 + \ldots + f(v_n)f_n$$

Proof: *Here $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of a vector basis for V and $B^* = \{f_1, f_2, \ldots, f_n\}$ be a basis for V^**
$\therefore B$ is linearly independent and $[B] = V$ and
B^* is linearly independent and $[B^*] = V^*$

(i) Let

$$
\begin{aligned}
&v \in V \\
&v \in V = [B] \\
&v \in [B] \\
&v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_v, \quad \forall\, \alpha_i \in \mathbb{R}, \quad i = 1, 2, \ldots n
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
f_1(v) &= f_1(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_v) \\
&= f_1(\alpha_1 v_1) + f_1(\alpha_2 v_2) + \ldots + f_1(\alpha_n v_n) \\
&= \alpha_1(f_1(v_1)) + \alpha_2(f_1(v_2)) + \ldots + \alpha_n(f_1(v_n)) \\
&= \alpha_1(1) + \alpha_2(0) + \ldots + \alpha_n(0) \\
f_1(v) &= \alpha_1
\end{aligned}
$$

Similarly we can prove that,

$$
\begin{aligned}
f_2(v) &= \alpha_2 \\
f_3(v) &= \alpha_3 \\
&\vdots \\
f_{1n}(v) &= \alpha_n
\end{aligned}
$$

Substitute this values in equation (19) we get,

$$v = f_1(v)v_1 + f_2(v)v_2 + \ldots + f_n(v)v_n$$

(ii) Let

$$
\begin{aligned}
&f \in V^* \\
&f \in V^* = [B^*] \\
&f \in [B^*] \\
&f = \alpha_1 f_1 + \alpha_2 f_2 \ldots + \alpha_n f_n, \quad \forall\, \alpha_i \in \mathbb{R}, \quad i = 1, 2, \ldots n
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
f(v_1) &= (\alpha_1 f_1 + \alpha_2 f_2 \ldots + \alpha_n f_n)(v_1) \\
&= (\alpha_1 f_1)(v_1) + (\alpha_2 f_2)(v_1) + \ldots + (\alpha_n f_n)(v_1) \\
&= \alpha_1(f_1(v_1)) + \alpha_2(f_2(v_1)) + \ldots + \alpha_n(f_n(v_1)) \\
&= \alpha_1(1) + \alpha_2(0) + \ldots + \alpha_n(0) \\
f(v_1) &= \alpha_1
\end{aligned}
$$

Similarly we can prove that,

$$
\begin{aligned}
f(v_2) &= \alpha_2 \\
f(v_3) &= \alpha_3 \\
&\vdots \\
f(v_n) &= \alpha_n
\end{aligned}
$$

Substitute this values in equation (20) we get,

$$f = f(v_1)f_1 + f(v_2)f_2 + \ldots + f(v_n)f_n$$

3 Operator Equations

Definition 3.1 *Let $T : U \to V$ be a linear map and the solution of the equation*

$$T(u_0) = v_0 \quad where \;\; v_0 \; is \; a \; fixed \; vector \; in \; V, \tag{21}$$

then the equation (21) is called an Operator Equation.

Definition 3.2 *If $v_0 = \overline{0}_V$, then the set of solution of the equation*

$$T(u) = \overline{0}_V \tag{22}$$

Then equation (22) is called homogeneous (H) equation and solution of this equation is called kernel of T.

Definition 3.3 *If $v_0 \neq \overline{0}_V$ then equation (21) is called non-homogeneous (NH) equation.*

Theorem 6 *Let $T : U \to V$ be a linear map.Given $v_0 \neq \overline{0}_V$ in V, the non-homogeneous equation*

$$(NH) \qquad T(u) = v_0$$

and the associated homogeneous equation

$$(H) \qquad T(u) = \overline{0}_V$$

have the following properties:
(a) If $v_0 \in R(T)$ and (H) has the trivial solution, namely $u = \overline{0}_U$ as its only solution, then (NH) has a unique solution.
(b) If $v_0 \in R(T)$ and (H) has a nontrivial solution, namely a solution $u \neq \overline{0}_U$, then (NH) has an infinite number of solutions.In this case if u_0 is a solution of (NH) is the linear variety $u_0 + K$, where $K = N(T)$ is the set of all solutions of (H).

Proof: *(a) If $v_0 \in R(T)$, then $T(u) = v_0$ has a solution.*
If $T(u) = \overline{0}_V$ has only one solution $u = \overline{0}_U$.
Then $N(T) = \{\overline{0}_U\}$
so, T is one-one.
This means $T(u) = v_0$ cannot more than one solution.
i.e. the solution of (NH) is unique.

(b) If $T(u) = \overline{0}_V$ has a nonzero solution, then $N(T) \neq \{\overline{0}_U\}$.
Let $u_0 \in U$ be a solution of (NH). It exists because $v_0 \in R(T)$.
Then $T(u_0) = v_0$.
Now if $u_k \in N(T)$, then

$$\begin{aligned} T(u_0 + u_k) &= T(u_0) + T(u_k) \quad (\because \; T \; is \; linear) \\ &= v_0 + \overline{0}_V \\ &= v_0 \end{aligned}$$

Therefore, $u_0 + u_k$ is a solution of (NH).
This is true for every $u_k \in N(T)$, and $N(T)$ has infinite number of elements.
So, (NH) has infinite number of solutions.

From above discussion it is obvious that $u_0 + K$, where $K = N(T)$, is contained in the solution set of (NH).
Conversely, if w be any other solution of (NH), then

$$T(w) = v_0 = T(u_0)$$

then

$$\begin{aligned} T(w) - T(u_0) &= \overline{0}_V \\ T(w - u_o) &= \overline{0}_V \\ w - u_0 &\in N(T) = K \\ w &\in u_0 + K \end{aligned}$$

Thus, the solution set of (NH) is precisely $u_0 + K$.

Example 5 *Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear map defined by*
$T(e_1) = \frac{1}{2} f_1,\ \ T(e_2) = \frac{1}{2} f_1,\ \ T(e_3) = f_2,\ \ T(e_4) = f_2,\ \ T(e_5) = \overline{0}.$
where $\{e_1, e_2, e_3, e_4, e_5\}$ is the standard basis for $\mathbb{R}^5$ and $\{f_1, f_2, f_3\}$ is the standard basis for $\mathbb{R}^3$.Then solve the equation

$$T(u) = (1, 1, 0)$$

Solution: *First we calculate the value of $T(x_1, x_2, x_3, x_4, x_5)$:*

$$\begin{aligned} T(x_1, x_2, x_3, x_4, x_5) &= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) + x_4 T(e_4) + x_5 T(e_5) \\ &= x_1 \frac{1}{2} f_1 + x_2 \frac{1}{2} f_1 + x_3 f_2 + x_4 f_2 + x_5 \overline{0} \\ &= \frac{x_1}{2}(1, 0, 0) + \frac{x_2}{2}(1, 0, 0) + x_3(0, 1, 0) + x_4(0, 1, 0) + x_5(0, 0, 0) \\ &= (\frac{x_1 + x_2}{2}, x_3 + x_4, 0) \end{aligned}$$

The associated homogeneous equation is:

$$\begin{aligned} T(x_1, x_2, x_3, x_4, x_5) &= \overline{0} \\ (\frac{x_1 + x_2}{2}, x_3 + x_4, 0) &= (0, 0, 0) \\ \frac{x_1 + x_2}{2} &= 0, \quad x_3 + x_4 = 0 \end{aligned}$$

we get, $x_2 = -x_1, x_3 = -x_4$

Thus, the kernel of T is the set of all vectors of the form $(x_1, -x_1, x_3, -x_3, x_5)$

i.e. $x_1(1, -1, 0, -, 0) + x_3(0, 0, 1, -1, 0) + x_5(0, 0, 0, 0, 1)$. *Hence*

$$N(T) = [(1, -1, 0, 0, 0), (0, 0, 1, -1, 0), (0, 0, 0, 0, 1)]$$

One particular solution of $T(u) = (1, 1, 0)$ is $u_0 = (2, 0, 1, 0, 0)$, which is obtained by putting $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 0$.
So the complete solution of the equation

$$T(u) = (1, 1, 0)$$

is the linear variety (2,0,1,0,0) + N(T), i.e. the set

$$(2, 0, 1, 0, 0) + \{(a, -a, b, -b, c)/a, b, c \ are \ real \ numbers \ \}$$

4 Annihilators:

Definition 4.1 *Let V be a real vector space and S be a non-empty subset of a vector space V, then the set $\{f \in V^*/f(x) = 0, \; \forall \; x \in S\}$ is called an annihilators of a set S and it is denoted by S^0.*

$$S^0 = \{f \in V^*/f(x) = 0, \; \forall \; x \in S\}$$

Theorem 7 *Let S be a non-empty subset of a vector space V, then prove that S^0 is a subspace of V^*.*

Proof: *Here S is a non-empty subset of a vector space V.*
let V be a real vector space and V^* be a dual space of a vector space V.

$$S^0 = \{f \in V^*/f(x) = 0, \; \forall \; x \in S\}$$
$$\overline{0}(x) = 0, \;\; \forall \; x \in S$$
$$\overline{0} \in S^0$$
$$S^0 \neq \phi$$

(i) Let $f_1, f_2 \in S^0$,we have to prove that $f_1 + f_2 \in S^0$
Here $f_1, f_2 \in S^0$
So $f_1(x) = 0, \;\; f_2(x) = 0, \;\; \forall \; x \in S$

$$\begin{aligned}(f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ &= 0 + 0 \\ &= 0 \\ f_1 + f_2 &\in S^0\end{aligned}$$

(ii) Let α be a scalar and let $f \in S^0$ then to prove that $\alpha f \in S^0$.
Here $f \in S^0$ so $f(x) = 0, \;\; \forall \; x \in S$

$$\begin{aligned}((\alpha f)(x) &= \alpha f(x) \\ &= \alpha 0 \\ &= 0 \\ \alpha f &\in S^0\end{aligned}$$

So from (i) and (ii) S^0 is a subspace of dual space of V^*.

Note: If $S = \overline{0}$ then $S^0 = V^*$.

5 Bilinear form

Definition 5.1 *Let V be a real vector space, a bilinear form $f : V \times V \to \mathbb{R}$ is a function of two variables such that,*
$\forall \; x, y, z \in V \;$ *and* $\alpha, \beta \in \mathbb{R}$

1. $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$
2. $f(x, \alpha y + \beta z) = \alpha f(x, y) + \beta f(x, z)$

Example 6 *If $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $f(x, y) = x_1 y_2 - x_2 y_1$ then prove that f is a bilinear form.*

Solution: *Here $\mathbb{R}^2$ is a vector space.*
Here $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x_1 y_2 - x_2 y_1$$

we have to prove f is a bilinear form.

(i) Take $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$
we have to prove $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$

$$\begin{aligned}
f(\alpha x + \beta y, z) &= f(\alpha(x_1, x_2) + \beta(y_1, y_2), (z_1, z_2)) \\
&= f((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2), (z_1, z_2)) \\
&= (\alpha x_1 + \beta y_1) z_2 - (\alpha x_2 + \beta y_2) z_1 \\
&= \alpha x_1 z_2 + \beta y_1 z_2 - \alpha x_2 z_1 - \beta y_2 z_1 \\
&= \alpha(x_1 z_2 - x_2 z_1) + \beta(y_1 z_2 - y_2 z_1) \\
&= \alpha f(x, z) + \beta f(y, z)
\end{aligned}$$

(ii) Take $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$
we have to prove $f(x, \alpha y + \beta z) = \alpha f(x, y) + \beta f(x, z)$

$$\begin{aligned}
f(x, \alpha y + \beta z) &= f((x_1, x_2), \alpha(y_1, y_2) + \beta(z_1, z_2)) \\
&= f((x_1, x_2), (\alpha y_1 + \beta z_1, \alpha y_2 + \beta z_2)) \\
&= x_1(\alpha y_2 + \beta z_2) - x_2(\alpha y_1 + \beta z_1) \\
&= \alpha x_1 y_2 + \beta x_1 z_2 - \alpha x_2 y_1 - \beta x_2 z_1 \\
&= \alpha(x_1 y_2 - x_2 y_1) + \beta(x_1 z_2 - x_2 z_1) \\
&= \alpha f(x, y) + \beta f(x, z)
\end{aligned}$$

From (i) and (ii) f is a bilinear form.

6 Exercises:

1. Find the dual basis corresponding to a basis $B = \{(1, -1, 1), (1, 1, -1), (-1, 1, 1)\}$ of $\mathbb{R}^3$.

2. Find the dual basis corresponding to a basis $B = \{(1, -2, 1), (-2, 0, 1), (0, 0, 1)\}$ of $\mathbb{R}^3$

3. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1.x_2, x_3) = (x_1 + x_2, x_1 - x_3)$, then solve the operator equation $T(x_1.x_2, x_3) = (6, 3)$.

4. If $x = (x_1, x_2, x_3) \in \mathbb{R}^3, y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $f(x, y) = x_1 y_2 - 3x_2 y_3 + x_3 y_1$, then prove that f is bilinear form.

5. If $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $f(x, y) = (x_1 - y_1)^2 + x_2 y_2$. Is f is bilinear form on $\mathbb{R}^2$.