SHRI GOVIND GURU UNIVERSITY, GODHRA
B.Sc. Semester-III
Mathematics-201
(ADVANCED CALCULUS-I)

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Outline

- **Topic-1:** Function of Several Variables.
- **Topic-2:** Evaluate the Limit by Definition.
- **Topic-3:** Limit of Function of Several Variables.
- **Topic-4:** Continuity of Function of Several Variables.

Introduction to Function of Several Variables

Definition (વ્યાખ્યા): A function $f: S \to R, S \subset R^n, n \in N - \{1\}$, R is the set of real numbers, is called a real valued function of n variables, where for each $x = \{x_1, x_2, x_3, ..., x_n\} \in S$, the value of f is a real number $f(x_1, x_2, x_3, ..., x_n)$.

Open Set: A set $S \subset R^n$ is said to be an open set, if every point of s is an interior point of S.

Closed Set: A set $S \subset R^n$ containing its all limit points is known as closed set.

Derived Set: A set of all limit points of set $S \subset R^n$ is known as derived set of S.

Introduction to Function of Several Variables

Open Region: A region which is consisting its interior points only is known as open region.

Closed Region: A region which is consisting its all interior points and boundary points is known as closed region.

Limit of function of two variables: Let $z = f(x, y)$ be any real valued function of two variables defined in a deleted δ-neighbourhood of point (x_0, y_0), and the limit of $f(x, y)$ as $(x, y) \rightarrow (x_0, y_0)$ is $l \in R$. If $\forall \epsilon > 0, \exists \delta > 0$ such that $0 < |(x, y) - (x_0, y_0) < \delta| \Rightarrow |f(x, y) - l| < \epsilon$, then l is called the limit of function $f(x, y)$. It is denoted as

$$\lim_{(x,y)\rightarrow(x_0,y_0)} f(x, y) = l.$$

Examples

Example 1: Evaluate the limit by definition if exist

$$\lim_{(x,y)\to(2,3)} [3xy] .$$

Solution: We have $f(x,y) = 3xy$ and $(x,y) \to (2,3)$,
$\Rightarrow 0 < |(x,y) - (2,3)| < \delta,$
$\Rightarrow 0 < |(x-2)| < \delta, 0 < |(y-3)| < \delta,$
$\Rightarrow -\delta < (x-2) < \delta, -\delta < (y-3) < \delta,$
$\Rightarrow 2-\delta < x < 2+\delta, 3-\delta < y < 3+\delta,$
$\Rightarrow 3(2-\delta)(3-\delta) < 3xy < 3(2+\delta)(3+\delta),$
$\Rightarrow 18 - (15\delta + 3\delta^2) < 3xy < 18 + (15\delta + 3\delta^2),$
$\Rightarrow 18 - \epsilon < 3xy < 18 + \epsilon,$
$\therefore 15\delta + 3\delta^2 = \epsilon,$
$\Rightarrow 3\delta^2 + 15\delta - \epsilon = 0,$

Examples

$\Rightarrow \delta = -\frac{5}{2} + \sqrt{\frac{\epsilon}{3} + \frac{25}{4}}$,
$\Rightarrow \delta > 0$.
Thus, by the definition of limit, we get
$\Rightarrow |3xy - 18| < \epsilon$,
$\Rightarrow \lim_{(x,y)\to(2,3)} [3xy] = 18$.

Example 2: Evaluate the limit by definition if exist

$$\lim_{(x,y)\to(2,1)} \frac{2x+y}{3y-x}.$$

Solution: We have $f(x,y) = \frac{2x+y}{3y-x}$ and $(x,y) \to (2,1)$,
$\Rightarrow 0 < |(x,y) - (2,1)| < \delta$,
$\Rightarrow 0 < |(x-2)| < \delta, 0 < |(y-1)| < \delta$,

Examples

$\Rightarrow -\delta < (x-2) < \delta, -\delta < (y-1) < \delta,$

$\Rightarrow 2-\delta < x < 2+\delta, 1-\delta < y < 1+\delta,$

$\Rightarrow 4-2\delta < 2x < 4+2\delta, 3-3\delta < 3y < 3+3\delta,$

$\Rightarrow 5-3\delta < 2x+y < 5+3\delta$ and $1-2\delta < 3y-x < 1+2\delta,$

Now, $1-2\delta < 3y-x < 1+2\delta$ can be written as

$\Rightarrow \frac{1}{1+2\delta} < \frac{1}{3y-x} < \frac{1}{1-2\delta},$

$\Rightarrow \frac{5-3\delta}{1+2\delta} < \frac{2x+y}{3y-x} < \frac{5+3\delta}{1-2\delta},$

$\Rightarrow 5-\frac{13\delta}{1+2\delta} < \frac{2x+y}{3y-x} < 5+\frac{13\delta}{1-2\delta},$

$\because \frac{13\delta}{1-2\delta} < \frac{13\delta}{1+2\delta},$

$\Rightarrow 5-\frac{13\delta}{1-2\delta} < \frac{2x+y}{3y-x} < 5+\frac{13\delta}{1-2\delta},$

$\Rightarrow 5-\epsilon < \frac{2x+y}{3y-x} < 5+\epsilon,$

Theorem

$\therefore \epsilon = \frac{13\delta}{1-2\delta}$,
$\Rightarrow \delta > 0$ as $\epsilon > 0$.
Thus, by the definition of limit, we get
$\Rightarrow |\frac{2x+y}{3y-x} - 5| < \epsilon$,
$\Rightarrow \lim_{(x,y)\to(2,1)} \dfrac{2x+y}{3y-x} = 5.$

Theorem 1: Let function $\phi(x)$ is continuous at a point $(a, \phi(a)) = (a, b)$ and $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and is equal to $l \in R$, then $\lim_{x\to a} f(x, \phi(x))$ exists and is equal to l.
Proof: We have

$$\lim_{(x,y)\to(a,b)} f(x, y) = l$$

$\therefore$ for $\epsilon > 0, \exists\delta > 0$ such that $|f(x, y) - l| < \epsilon$ whenever

Theorem

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1 \tag{1}$$

$\because \phi(x)$ is continuous at $x = a$,
$\therefore$ Corresponding to $\delta_1, \exists \delta_2 > 0$ such that

$$|\phi(x) - \phi(a)| < \frac{\delta_1}{\sqrt{2}}, |x-a| < \delta_2 \tag{2}$$

Let $\delta = min\left\{\frac{\delta_1}{\sqrt{2}}, \delta_2\right\}$,
$\therefore |x-a| < \delta$,
$\Rightarrow |x-a| < \frac{\delta_1}{\sqrt{2}}, |x-a| < \delta_2$,
$\Rightarrow |x-a| < \frac{\delta_1}{\sqrt{2}}|, |\phi(x) - \phi(a)| < \frac{\delta_1}{\sqrt{2}}$. Thus,

$$|x-a| < \delta \Rightarrow \sqrt{(x-a)^2 + (\phi(x) - \phi(a))^2} < \delta_1 \tag{3}$$

Definition

$\therefore$ equation (1) and (3) ,
$\Rightarrow |x-a| < \delta,$
$\Rightarrow |f(x,y) - l| < \epsilon,$
$\Rightarrow |f(x,\phi(x)) - l| < \epsilon,$ thus we get,

$$\lim_{x\to a} f(x,\phi(x)) = l, \epsilon > 0.$$

Existence of limits: Let $y = \phi(x)$ and $y = \psi(x)$ are two functions which are continuous at the point $x = a$ and if $\lim_{x\to a} f(x,\phi(x)) = \lim_{x\to a} f(x,\psi(x))$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists. Further, if $\lim_{x\to a} f(x,\phi(x)) \neq \lim_{x\to a} f(x,\psi(x))$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Examples

Example 1: Evaluate the limit if exists,

$$\lim_{(x,y)\to(0,0)} f(x,y), \textit{ where}$$

$$f(x,y) = \tan^{-1}\left(\tfrac{y}{x}\right), x \neq 0, y \neq 0$$
$$= 0, \qquad x = 0, y = 0.$$

Solution: Let $y = \phi(x) = x$ and $y = \psi(x) = x^2$ are two functions which are continuous at point $x = 0$. First, we take $y = \phi(x) = x$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, \phi(x)),$$

$$= \lim_{x\to 0} f(x,x),$$

$$= \lim_{x\to 0} \tan^{-1}\left(\frac{x}{x}\right),$$

Examples

$$= \frac{\pi}{4}.$$

Further, we take $y = \psi(x) = x^2$,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0} f(x, \psi(x)) \\
&= \lim_{x\to 0} f\left(x, x^2\right), \\
&= \lim_{x\to 0} \tan^{-1}\left(\frac{x^2}{x}\right), \\
&= \lim_{x\to 0} \tan^{-1}(x), \\
&= 0.
\end{aligned}$$

Examples

Thus, $\lim_{x \to 0} f(x, \phi(x)) \neq \lim_{x \to 0} f(x, \psi(x))$,
$\Rightarrow$ Limit does not exist.
Another Solution: Let $y = \phi_i(x) = m_i x$,

$$\Rightarrow f(x, y) = f(x, \phi_i(x)),$$

$$\Rightarrow f(x, y) = f(x, m_i x),$$

$$\Rightarrow f(x, y) = \tan^{-1}(m_i),$$

$$\therefore \lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} f(x, \phi(x)),$$

$$= \lim_{x \to 0} \tan^{-1} m_i,$$

which depends on m_i.

Examples

$\therefore \lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{y}{x}\right)$ does not exist.

Example 2: Evaluate the limit if exists,

$$\lim_{(x,y)\to(0,0)} f(x,y), \textit{ where}$$

$$\begin{aligned} f(x,y) &= \tfrac{x^3-y^3}{x^2+y^2}, x \neq 0, y \neq 0 \\ &= 0, \qquad x = 0, y = 0. \end{aligned}$$

Solution: Let $y = \phi(x) = x$ and $y = \psi(x) = x^2$ are two functions which are continuous at point $x = 0$. First, we take $y = \phi(x) = x$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, \phi(x)),$$

$$= \lim_{x\to 0} f(x,x),$$

Examples

$$= \lim_{x \to 0} \frac{x^3 - x^3}{x^2 + x^2},$$
$$= 0.$$

Further, we take $y = \psi(x) = x^2$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, \psi(x)),$$
$$= \lim_{x\to 0} f\left(x, x^2\right),$$
$$= \lim_{x\to 0} \frac{x^3 - x^6}{x^2 + x^6},$$
$$= 0.$$

Thus, $\lim_{x\to 0} f(x, \phi(x)) = \lim_{x\to 0} f(x, \psi(x)) \Rightarrow$ Limit exists.

Continued...

Iterated Limits

Statement: Let $f : R^2 \rightarrow R$ be a function and $(a, b) \in R^2$ be limit point. If $\lim_{x \rightarrow a} \left\{ \lim_{y \rightarrow b} f(x, y) \right\}$ and $\lim_{y \rightarrow b} \left\{ \lim_{x \rightarrow a} f(x, y) \right\}$ exist, then these limits are called **iterated limits**.

Remarks:

[1]. Iterated limits may or may not exist.

[2]. If exist, then iterated limits may or may not equal.

[3]. If limit and iterated limits exist then they are all equal.

Example 1: Find $\lim_{x \rightarrow a} \left\{ \lim_{y \rightarrow b} f(x, y) \right\}$ and $\lim_{y \rightarrow b} \left\{ \lim_{x \rightarrow a} f(x, y) \right\}$ if

$$\begin{aligned} f(x, y) &= \frac{x^2 - y^2}{x^2 + y^2}, x \neq 0, y \neq 0 \\ &= 2, \qquad x = 0, y = 0. \end{aligned}$$

Examples

Solution: We have,

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}\right\},$$

$$= \lim_{x\to 0}\left\{\frac{x^2}{x^2}\right\},$$

$$= 1,$$

$$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}\right\},$$

$$= \lim_{y\to 0}\left\{\frac{-y^2}{y^2}\right\},$$

$$= -1,$$

Examples

Thus, we get

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} \neq \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}.$$

Example 2: Find $\lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\}$ and $\lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\}$ if

$$\begin{aligned} f(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}, x\neq 0, y\neq 0 \\ &= 2, \qquad\qquad x=0, y=0. \end{aligned}$$

Solution: We have,

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{xy}{\sqrt{x^2+y^2}}\right\},$$

Examples

$$= \lim_{x\to 0} \left\{ \frac{0}{\sqrt{x^2}} \right\},$$

$$= 0,$$

$$\lim_{y\to 0} \left\{ \lim_{x\to 0} f(x,y) \right\} = \lim_{y\to 0} \left\{ \lim_{x\to 0} \frac{xy}{\sqrt{x^2+y^2}} \right\},$$

$$= \lim_{y\to 0} \left\{ \frac{0}{\sqrt{y^2}} \right\},$$

$$= 0,$$

Thus, we get

$$\lim_{x\to 0} \left\{ \lim_{y\to 0} f(x,y) \right\} = \lim_{y\to 0} \left\{ \lim_{x\to 0} f(x,y) \right\}.$$

Continuity

Statement: A function $f(x, y)$ is **continuous** at a point (a, b) in its domain if the following conditions are satisfied:

1. Function $f(x, y)$ is defined at point $f(a, b)$ *i.e.* $f(a, b)$ exists,

2. $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists, and

3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example 1: Discuss the continuity of the function $f(x, y)$ at point $(0, 0)$,

$$\begin{aligned} f(x, y) &= \frac{x^2-y^2}{x+y}, x \neq 0, y \neq 0 \\ &= 0, \qquad x = 0, y = 0. \end{aligned}$$

Solution: Let $y = \phi(x) = x$ and $y = \psi(x) = x^2$ are two functions. First, we take $y = \phi(x) = x$,

Continued...

Examples

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, \phi(x)),$$
$$= \lim_{x\to 0} f(x,x),$$
$$= \lim_{x\to 0} \frac{x^2 - x^2}{x + x},$$
$$= 0.$$

Further, we take $y = \psi(x) = x^2$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, \psi(x)),$$
$$= \lim_{x\to 0} f\left(x, x^2\right),$$

Examples

$$= \lim_{x \to 0} \frac{x^2 - x^4}{x + x^2},$$
$$= 0.$$

$\therefore \lim_{x \to 0} f(x, \phi(x)) = \lim_{x \to 0} f(x, \psi(x)) = 0.$
$\because f(0,0) = 0,$
$\Rightarrow \lim_{(x,y) \to (0,0)} f(x,y) = 0 = f(0,0).$
Thus, the function $f(x,y)$ is continuous at point $(0,0)$.
Example 2: Discuss the continuity of the function $f(x,y)$ at point $(0,0)$,

$$f(x,y) = \frac{xy^2}{x^2+y^2}, \quad x \neq 0, y \neq 0$$
$$= 0, \quad x = 0, y = 0.$$

Continued...

Examples

Solution: Let $y = \phi(x) = x$ and $y = \psi(x) = x^2$ are two functions. First, we take $y = \phi(x) = x$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,\phi(x)),$$

$$= \lim_{x\to 0} f(x,x),$$

$$= \lim_{x\to 0} \frac{xx^2}{x^2 + x^2},$$

$$= 0.$$

Further, we take $y = \psi(x) = x^2$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,\psi(x)),$$

Examples

$$= \lim_{x \to 0} f\left(x, x^2\right),$$

$$= \lim_{x \to 0} \frac{xx^4}{x^2 + x^4},$$

$$= 0.$$

$\therefore \lim_{x \to 0} f(x, \phi(x)) = \lim_{x \to 0} f(x, \psi(x)) = 0.$

$\because f(0, 0) = 0,$

$\Rightarrow \lim_{(x,y) \to (0,0)} f(x, y) = 0 = f(0, 0).$

Thus, the function $f(x, y)$ is continuous at point $(0, 0)$.

Examples

Example 3: Discuss the continuity of the function $f(x,y)$ at point $(0,0)$,

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & x \neq 0, y \neq 0 \\ = 2, & x = 0, y = 0. \end{cases}$$

Solution: Let $y = \phi(x) = x$ and $y = \psi(x) = x^2$ are two functions. First, we take $y = \phi(x) = x$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,\phi(x)),$$

$$= \lim_{x\to 0} f(x,x),$$

$$= \lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2},$$

$$= 0.$$

Examples

Further, we take $y = \psi(x) = x^2$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, \psi(x)),$$
$$= \lim_{x\to 0} f\left(x, x^2\right),$$
$$= \lim_{x\to 0} \frac{x^2 - x^4}{x^2 + x^4},$$
$$= 1.$$

$\therefore \lim_{x\to 0} f(x, \phi(x)) \neq \lim_{x\to 0} f(x, \psi(x))$.
$\because f(0,0) = 2$,
$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$.
Thus, the function $f(x,y)$ is discontinuous at point $(0,0)$.

Bibliography I

References

[1]. **Advanced Calculus-I**, 2014-2015 edition, **Nirav Prakashan**, Ahmedabad, Gujarat.

THANK YOU