SHRI GOVIND GURU UNIVERSITY
B.Sc.Sem-5 Material
BSC0C506D:Mathematics(Theory)
Operation Research -1

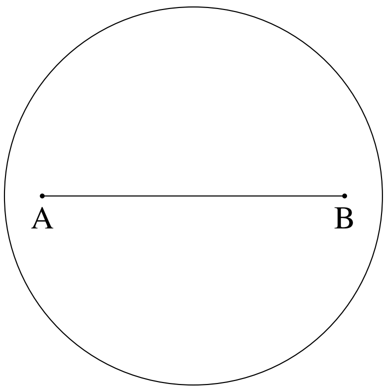
Unit-I

Unit-I:Convex set and Linear Programming Problem
Convex set,Extreme Points of convex set,convex combination,Examples of convex sets and Theorems on convexity,Formulation techniques of LP problems (Only Examples)

1 Convex set

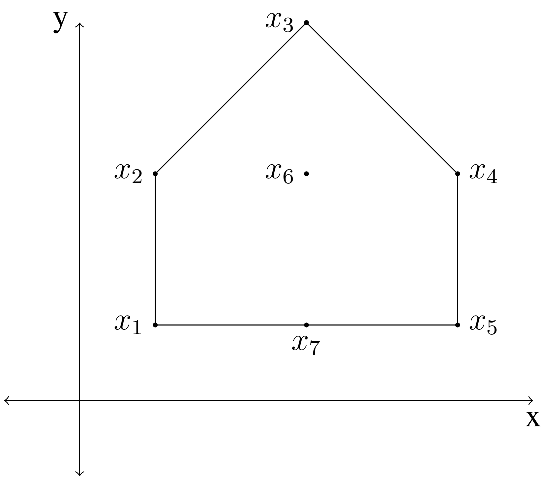
Definition 1.1 *A subset S of E^n is said to be convex set if for all pairs $x_1, x_2 \in S$ any linear combination $\theta x_1 + (1-\theta)x_2, \;\; (0 \le \theta \le 1)$ is also contained in S.*

In other words a convex set is a set of points such that given any two points A, B *in that set the line AB joining them lies entirely within that set.*



Convex set

Definition 1.2 *A point x in a convex set S is called an extreme point or vertex of S it does not lie on any line segment joining any two distinct points say x_1 and x_2 in S.*



Extreme and Non-extreme Points

Above figure represents some examples of extreme and non-extreme points. It may be noted that points $x_1, x_2, x_3, x_4 \ \ and \ x_5$ are extreme points of convex set S, where x_6 and x_7 are non-extreme points.

Example 1 *Show that the unit ball in E^n is convex set.*

Solution: Here unit ball in E^n is defined as,

$$S = \{u \in E^n, \ \|u\| \leq 1\}$$

Let $x, y \in S$, so $\|x\| \leq 1$ *and* $\|y\| \leq 1$.
Now,

$$\begin{aligned} \|\theta x + (1-\theta)y\| &\leq \|\theta x\| + \|(1-\theta)y\| \ \ (\because \ by \ triangle \ inequality) \\ &\leq \theta\|x\| + (1-\theta)\|y\| \\ &\leq \theta + (1-\theta) \\ &\leq 1 \end{aligned}$$

Thus, $\theta x + (1-\theta)y \in S$

Hence S is a convex set.

Example 2 *Show that $S = \{(x_1, \ x_2); 2x_1 + 3x_2 = 7\} \in \mathbb{R}^2$ is a convex set.*

Solution: Let $x, y \in S$

$$x = (x_1, \ x_2), \ y = (y_1, \ y_2) \in S$$

$$\therefore \ 2x_1 + 3x_2 = 7, \ \ 2y_1 + 3y_2 = 7 \tag{1}$$

Now we have to prove $\theta x + (1 - \theta y) \in S$

$$\begin{aligned} \theta x + (1-\theta)y &= \theta(x_1, \ x_2) + (1-\theta)(y_1, \ y_2) \\ &= (\theta x_1, \ \theta x_2) + ((1-\theta)y_1, \ (1-\theta)y_2) \\ &= (\theta x_1 + (1-\theta)y_1, \ \theta x_2 + (1-\theta)y_2) \\ &= 2(\theta x_1 + (1-\theta)y_1) + 3(\theta x_2 + (1-\theta)y_2) \\ &= 2(\theta x_1) + 2(1-\theta)y_1 + 3(\theta x_2) + 3(1-\theta)y_2 \\ &= \theta(2x_1 + 3x_2) + (1-\theta)(2y_1 + 3y_2) \\ &= 7\theta + (1-\theta)7 \qquad (From \ \ equation \ \ (1)) \\ &= 7\theta + 7 - 7\theta \\ &= 7 \end{aligned}$$

Thus, $\theta x + (1 - \theta y) \in S$

Hence S is a convex set.

Example 3 *Prove that $B = \{(x_1,\ x_2)/{x_1}^2 + {x_2}^2 \leq 4\}$ is convex set.*

Solution: Let $x, y \in B$

$$x = (x_1,\ x_2),\ y = (y_1,\ y_2) \in B$$

$$\therefore\ {x_1}^2 + {x_2}^2 \leq 4,\ \ {y_1}^2 + {y_2}^2 \leq 4 \tag{2}$$

Now we have to prove $\theta x + (1 - \theta y) \in B$

$$\begin{aligned}
\theta x + (1-\theta)y &= \theta(x_1,\ x_2) + (1-\theta)(y_1,\ y_2)\\
&= (\theta x_1,\ \theta x_2) + ((1-\theta)y_1,\ (1-\theta)y_2)\\
&= (\theta x_1 + (1-\theta)y_1, \theta x_2 + (1-\theta)y_2)\\
&= (\theta x_1 + (1-\theta)y_1)^2 + (\theta x_2 + (1-\theta)y_2)^2\\
&= \theta^2 {x_1}^2 + (1-\theta)^2 {y_1}^2 + 2\theta x_1(1-\theta)y_1 + \theta^2 {x_2}^2 + (1-\theta)^2 {y_2}^2 + 2\theta x_2(1-\theta)y_2\\
&= \theta^2({x_1}^2 + {x_2}^2) + (1-\theta)^2({y_1}^2 + {y_2}^2) + 2\theta(1-\theta)(x_1y_1 + x_2y_2)\\
&\leq 4\theta^2 + 4(1-\theta)^2 + 2\theta(1-\theta)4 \qquad (From\ \ equation\ \ (2))\\
&\leq 4\theta^2 + 4 + 4\theta^2 - 8\theta + 8\theta - 8\theta^2\\
&\leq 4
\end{aligned}$$

Thus, $\theta x + (1 - \theta y) \in B$

Hence B is a convex set.

Definition 1.3 *Let $u_1, u_2, \ldots, u_m$ be m points in E^n and let $\theta_1, \theta_2, \ldots, \theta_m$ be non-negative real numbers such that*

$$\sum_{j=1}^{m} \theta_j = 1$$

then $v = \theta_1 u_1 + \theta_2 u_2 + \ldots + \theta_m u_m$ is called the convex linear combination if point u_j, $j = 1, 2, \ldots, m$.

Theorem 1 *A necessary and sufficient condition for a set S to be convex is that every convex linear combination in S belongs to S.*

OR

Show that the set of all convex combinations of a finite numbers of points of $S \subset E^n$ is a convex.

OR

Prove that $K \subset \mathbb{R}^n$ is a convex set if and only if every convex linear combination of elements in K also belongs to K.

Proof:($\Rightarrow$) we assume that S consists m points say $u_1, u_2, \ldots, u_m$.
Suppose that any convex combination of $u_1, u_2, \ldots, u_m$ is also in S.
This implies any convex combination of two points in S belongs to S.
So by the definition of convex set S is convex set.

($\Leftarrow$) The proof of converse is done by induction on m, the number of elements in convex combination.
Let S be a convex set and

$$v = \sum_{j=1}^{m} \theta_j u_j$$

The theorem is true for $m = 1$ by default.
The theorem is true for $m = 2$ by the definition of convexity,
Now, we need only prove that theorem is true for $m = k + 1$, under the inductive hypothesis that it is true for $m = k$
Suppose that it is true for $m = k$.
Now consider

$$v = \sum_{j=1}^{m} \theta_j u_j, \quad \sum_{j=1}^{k+1} \theta_j = 1, \quad \theta_j \geq 0, \quad j = 1, 2, \dots k + 1$$

Here two cases arises:

Case-I:- $\theta_{k+1} = 0$

$$\therefore v = \sum_{j=1}^{k+1} \theta_j u_j = \sum_{j=1}^{k} \theta_j u_j \in S \qquad (by \quad hypothesis)$$

Case-II:- $\theta_{k+1} > 0$

$$v = \sum_{j=1}^{k+1} \theta_j u_j$$

$$v = (1 - \theta_{k+1}) \sum_{j=1}^{k} \frac{\theta_j}{1 - \theta_{k+1}} u_j + \theta_{k+1} u_{k+1}$$

$$since \quad \sum_{j=1}^{k+1} \theta_j = 1$$

$$\Rightarrow \sum_{j=1}^{k} \theta_j + \theta_{k+1} = 1$$

$$\Rightarrow \sum_{j=1}^{k} \theta_j = 1 - \theta_{k+1}$$

$$\Rightarrow \sum_{j=1}^{k} \frac{\theta_j}{1 - \theta_{k+1}} = 1$$

so the vector

$$z = \sum_{j=1}^{k} \frac{\theta_j}{(1 - \theta_{k+1})} u_j = 1$$

is a convex combination of the k vectors $u_1, u_2, \dots, u_k$.
so, by the induction hypothesis, z is in S.
By the definition of convexity

$$v = (1 - \theta_{k+1}) z + \theta_{k+1} u_{k+1}$$

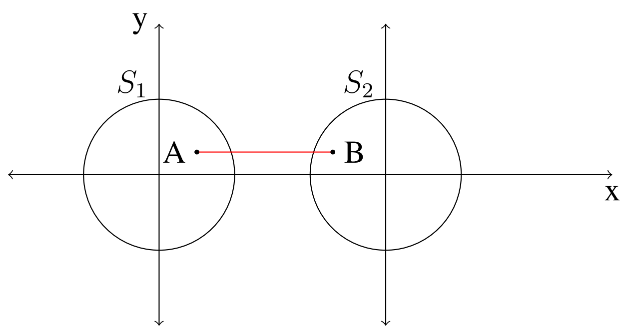
is in S.

Theorem 2 *Prove that the intersection of two convex sets is a convex set. Is union of two convex sets is necessarily convex set. Justify your answer.*

Proof: Let S_1 and S_2 are two convex sets.
We have to prove that $S_1 \cap S_2$ is also convex set.
Let $x, y \in S_1 \cap S_2$
$\Rightarrow x, y \in S_1$ *and* $x, y \in S_2$
$\Rightarrow \theta x + (1-\theta)y \in S_1$ *and* $\theta x + (1-\theta)y \in S_2$ ($\because S_1, S_2$ *are convex sets*)
$\Rightarrow \theta x + (1-\theta)y \in S_1 \cap S_2$
Thus $S_1 \cap S_2$ is a convex set.

Union of two convex sets is not necessarily convex set

Counter Example:-
Let $S_1 : x^2 + y^2 \leq 1$ is convex set and
Let $S_2 : (x-3)^2 + y^2 \leq 1$ is a convex set.
Now, we construct set S_1 and S_2 geometrically.



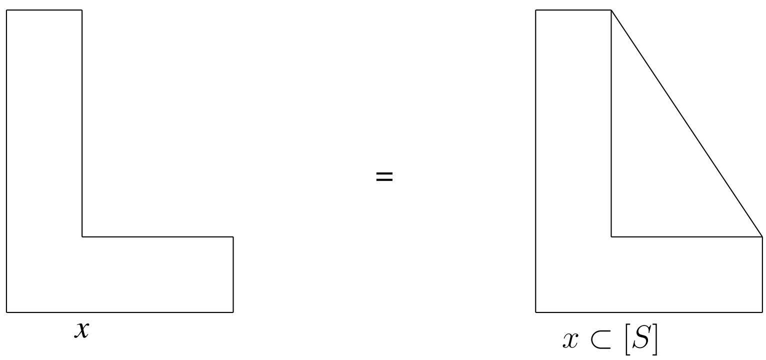
By above diagram it is clear that line AB does not belong to $S_1 \cup S_2$. so, $S_1 \cup S_2$ is not a convex set.

Definition 1.4 *Let S be a set of vectors in E^n then, the set of all convex combination of every finite subset of S is called the convex hull of S and it is written as $[S]$.*

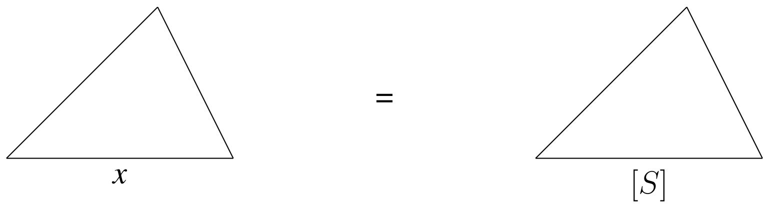
OR

A convex hull is the smallest convex set that contains x.

Example 4 *Convex hull of*



Example 5 *convex hull of*



Definition 1.5 *Let $a_1, a_2, \ldots, a_n$ denotes n vectors in E^n and let $b_1, b_2, \ldots, b_m$ denote the corresponding scalars*

$$let \;\; H_j = \{u_j \in E^n : a_i u_j \leq b_j\}$$

Each H_j is Half-Space. The set $S = \bigcap_{j=1}^{m} H_j$ is a polyhedral convex (convex polyhedral) set.

2 Formulation techniques of LP problems

A Linear Programming Problem (LPP) is a special case of a mathematical programming problem from an analytical perspective, a mathematical program tries to identify an extreme(i.e. minimum or maximum) point of a function $f(x_1, x_2, \ldots, x_n)$, which furthermore satisfies a set of constraints $g(x_1, x_2, \ldots, x_n) \geq b$

Definition 2.1 *Linear Programming is the specialization of mathematical programming to the case where both function f, to be called the objective function, and the problem constraint are linear.*

Model Components:- A model consists of linear relationships representing a firms objective and resource constraints.

Decision variables:- Mathematical symbols representing levels of activity of an operation.

Objective function:- A linear mathematical relationship describing and objective of the firm, in terms of decision variables, that is maximized or minimized.

Constraints:- Restrictions placed on the firm by the operating environment stated in linear relationships of the decision variables.

Parameters/ Cost Coefficients:- Numerical coefficient and constraints used in the objective function and constraints equations.

Thus, LPP is a collection of the objective function, the set of constraints and the set of non-negative constraints.

The general Mathematical Model of Linear Programming Problem:-

The general linear programming problem with n decision variables and m constraints can be stated in the following form:

Find values of decision variables $x_1, x_2, \ldots, x_n$ that optimize (maximize or minimize)

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to the linear constraints,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq) \ b_m \\ and \ \ x_1, x_2, \ldots, x_n \geq 0. \end{aligned}$$

The above formulation can also be expressed in a compact form using summation sign.

$$optimize \ (max. \ or \ min.) \ \ z = \sum_{j=1}^{n} c_j x_j \qquad (objective \ function)$$

subject to the linear constraints

$$\sum_{j=1}^{n} a_{ij}x_j \ (\leq, =, \geq) \ b_i; \ \ i = 1, 2, \ldots, m \quad (constraints)$$

$$and \ \ x_j \geq 0, \ \ j = 1, 2, \ldots n. \quad (non-negative \ \ conditions)$$

Steps of formulating of LPP :-

(i) Identify the decision variable of the problems.

(ii) Construct the objective function as a linear combination of decision variable.

(iii) Identify the constraints of the problem such as resources, limitation, inter relation between variables,etc. Formulate this constraints as linear equality or inequality in terms of non-negative decision variables.

Example 6 *A Manufacturer two types of models M and N. Each M model requires 4 hours of grinding and 2 hours of polishing whereas each N model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinder and 3 polisher. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on model M is Rs. 3 and model N is Rs. 4 whatever is produced in a week is sold in the market. How should the manufacture allocate his production capacity to the two types of models so that the maximize profit in a week ?*

Solution:- Let x_1 be the number of model M to be produced and let x_2 be the number of model N to be produced.

$$clearly, \qquad x_1, x_2 \geq 0$$

The manufacturer gets profit of Rs. 3 for model M and Rs. 4 for model N
So, the objective function is to maximize profit

$$maximize \quad z = 3x_1 + 4x_2$$

Each model M requires 4 hours of grinding and each model N requires 2 hours of grinding company has 2 grinder and each work 40 hours for week.

$$subject \ \ constraint \quad 4x_1 + 2x_2 \leq 80$$

Each model M requires 2 hours of polishing and each model N requires 5 hours of polishing company has 3 polisher and maximum available time for polishing is 60 hours for a week.

$$subject \ \ constraint \quad 2x_1 + 5x_2 \leq 180$$

$\therefore$ The manufacturer allocation problem to

$$maximize \quad z = 3x_1 + 4x_2$$

subject to the constraints,

$$4x_1 + 2x_2 \leq 80$$

$$2x_1 + 5x_2 \leq 180$$

$$and \quad x_1, x_2 \geq 0$$

Example 7 *A manufacturer produces two different models X and Y of the same product. Model X makes a contribution of Rs. 50 per unit and model Y Rs. 30 per unit towards total profit. Raw materials r_1 and r_2 are required for production at least 18 kg of r_1 and 12 kg of r_2 must be used daily also at most 34 hours of labour are to be utilized. A quantity of 2 kg of r_1 is needed for model X and 1 kg of r_1 for model Y. For each X and Y 1 kg of r_2 is required.It takes 3 hours to manufacture model X and 2 hour to manufacture model Y. A manufacturer wishes to maximize the profit formulate the linear programming problem.*

Solution:- Let x_1 be the number of model X to be produced and let x_2 be the number of model Y to be produced.

$$clearly, \quad x_1, x_2 \geq 0$$

Model X makes a contributing of Rs. 50 and Model Y makes a contributing of Rs. 30 so the objective function is to be maximize profit.

$$maximize \quad z = 50x_1 + 30x_2$$

For model X, 2 kg of r_1 is needed for model Y, 1 kg of r_1 is needed and 18 kg of r_1 must be used.

$$subject \;\; constraint \quad 2x_1 + x_2 \leq 18$$

For model X, 1 kg of r_2 is needed for model Y, 1 kg of r_2 is needed and 12 kg of r_2 must be used.

$$subject \;\; constraint \quad x_1 + x_2 \leq 12$$

Model X takes 3 hours to manufactured and model Y takes 2 hours to manufactured and 34 hours of labour are to be utilized. So,

$$subject \;\; constraint \quad 3x_1 + 2x_2 \leq 34$$

$\therefore$ The manufacturer maximize profit is

$$maximize \quad z = 50x_1 + 30x_2$$

subject to the constraints,

$$2x_1 + x_2 \leq 18$$

$$x_1 + x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 34$$

$$and \quad x_1, x_2 \geq 0$$

Example 8 *A person requires 10,12 and 12 units of chemicals A,B and C respectively for his garden. A liquid product contains 5,2 and 1 units of A,B and C respectively per jar A dry product contains 1,2 and 4 units of A,B and C respectively per carton. If the liquid product sells for Rs.30 per jar and the dry product sells for Rs 35 per carton to minimize the cost and meet the requirements ? Formulate the Linear Programming Problem.*

Solution:- Let x_1 be the liquid product contain and x_2 be the dry product contain

$$clearly, \quad x_1, x_2 \geq 0$$

Liquid product sells 30 Rs. per jar and dry product sells for 35 Rs. per carton
so, objective function is to minimized the cost

$$z = 30x_1 + 35x_2$$

A person required for chemical A, 5 units per jar and 1 unit per carton and required contain 10 units.

$$subject \ \ constraint \quad 5x_1 + x_2 \geq 10$$

A person required for chemical B, 2 units per jar and 2 units per carton and required contain 12 units.

$$subject \ \ constraint \quad 2x_1 + 2x_2 \geq 12$$

A person required for chemical C, 1 unit per jar and 4 units per carton and required contain 12 units.

$$subject \ \ constraint \quad x_1 + 4x_2 \geq 12$$

so, purchased in order to minimize case,

$$maximize \quad z = 30x_1 + 35x_2$$

subject to the constraints,

$$5x_1 + x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$x_1 + 4x_2 \geq 12$$
$$and \quad x_1, x_2 \geq 0$$

3 Exercises

1. Show that $S = \{(x_1, x_2); 3{x_1}^2 + 2{x_2}^2 \leq 6\}$ is a convex set.

2. Show that $S = \{(x_1, x_2, x_3); {x_1}^2 + {x_2}^2 + {x_3}^2 \leq 1\}$ is a convex set.

3. Show that $S = \{(x_1, x_2, x_3); 2x_1 - x_2 + x_3 \leq 4\}$ is a convex set.

4. If $S_1 = \{x \in E^n; \|x\| \leq 2\}$ and $S_2 = \{x \in E^n; \|x\| = 2\}$ determine the convexity of S_1 and S_2.

5. If $S_1 = \{x \in E^n; \|x\| \leq 1\}$ and $S_2 = \{x \in E^n; \|x\| = 1\}$ determine the convexity of S_1 and S_2.